MAGNETOHYDRODYNAMICS Vol. 40 (2004), No. 1, pp. 1–??

Non-linear waves in a thin layer of magnetic fluid

A. N. Golubiatnikov

Lomonosov Moscow State University, 119192, Moscow, Michurinsky Prospekt, 1

The problem of propagation of non-linear waves with small amplitude in a thin layer of paramagnetic liquid with variable thickness in heterogeneous magnetic field is considered in shallow water theory. The equations of motion are derived and the method of solving the problem with a weak discontinuity is developed. With the help of this method the problem with a weak shock wave is solved. The example of equilibrium initial state is considered.

1. Shallow water equations in magnetic field. We will consider the flat problem. Let $y = \zeta(x,t)$ and $y = -h(x,t)$ be the forms of higher and lower surfaces in relation to the constant gravity field g. The function $h(x,t)$ is given. As it known [1, 2], equations of motion in shallow water theory could be derived in the assumption of smallness of the ratio of layer thickness to the characteristic wavelength $\varepsilon = (\zeta + h)/\lambda \le 1$. Permeability μ is considered to be piecewise constant, $\mu = 1$ outside the layer and $\mu > 1$ within the layer. Thus, the motion of liquid and magnetic field are connected to each other only by boundary conditions on the surface of layer [3].

Vertically we solve the equation of equilibrium, and the solution defines the pressure

$$p = p_1(x,t) - \rho g (y - \zeta).$$

Normal to the surface of the layer, taking into account the order terms of the ε, is $\mathbf{n} = (-\zeta_x, 1)$, tangent vector is $\tau = (1, \zeta_x)$. Subscript x means derivative.

Pressure above the layer is assumed to be zero. Then an arbitrary function p_1, taking into account conditions $[H_\tau] = 0$ and $[\mu H_n] = 0$, is defined by the jump of normal-normal component of Maxwell stress tensor

$$p_1 = \frac{1}{8\pi}\left[\mu(H_n^2 - H_\tau^2)\right]_0^1 = -\frac{\mu - 1}{8\pi\mu}((H_n^0)^2 + \mu(H_\tau^0)^2) < 0. \tag{1}$$

The state outside the layer is denoted by zero, and the state into the layer is denoted by one. Surface tension is not considered. The behaviour analysis of magnetic field, that is regular in the neighborhood of the layer, shows that external field distortion by the thin layer is appears only in terms of order ε^2. It could be seen in example of flattened ellipsoid [4]. Thus, in the expression (1), that is calculated with the terms of order ε, only the external magnetic field enters. Let us stopped on the term of order 1.

$$p_1 = \frac{1}{8\pi}\left([1/\mu](H_2^0)^2 - [\mu](H_1^0)^2\right).$$

Traditional approach to the calculation of hydrodynamics terms in the theory of an ideal incompressible homogeneous fluid [1, 2] with considering of known variable depth $h(x,t)$ gives the following equations for the average to the thickness of layer velocity of longitudinal motion v and thickness of layer $\zeta + h$

$$(\zeta+h)_t+((\zeta+h)v)_x=0, \quad v_t+vv_x+g\,\zeta_x=-p_{1,x}/\rho, \tag{2}$$

where ρ is a density of liquid.

It is very helpful to introduce a mass of part of layer m, that is counted, for example, from movable wall (piston), as an independent lagrangian variable, which included in the function $x(t,m)$. So, the first equation in (2) gives us the following: $\rho(\zeta+h)=1/x_m$ and $v=x_t$. As a result we get

$$\rho x_{tt}+\left(g/(2x_m^2)\right)_m=\rho g h_x-p_{1,x}. \tag{3}$$

Equation (3) has the form of Euler equation with lagrangian

$$\Lambda=\frac{\rho x_t^2}{2}-\frac{g}{2x_m}+\rho g h(x,t)-p_1(x,t), \tag{4}$$

which will be used next in solving problem with a weak discontinuity.

Conditions on strong discontinuities (discontinuities of the first derivatives of the law of motion $x(t,m)$) in the absence of concentrated inflows of mass, momentum and energy follow from the form of Λ [3]. Let $t=T(m)$ be the movement time in mass of discontinuity. Let us introduce shock time $\tau=t-T(m)$, that is equals to zero on the discontinuity. Thus, in the variables τ,m we have

$$x_t=x_\tau, \quad x_m(t,m)=x_m(\tau,m)-T'(m)x_\tau.$$

It should be mentioned that in this variables operation of differentiation with respect to the m on the discontinuity saves the continuity of differentiable function.

On the strong discontinuity we have

$$[x]=0, \quad [\Lambda_{x_\tau}]=0. \tag{5}$$

In addition to that conditions we should write one more inequality, that is connected with the energy dissipation in the shallow water theory [1],

$$[x_\tau\Lambda_{x_\tau}-\Lambda]_0^1\le 0.$$

The state 0 is correspond to the state before discontinuity and the state 1 is correspond to the state after discontinuity. Function $T(m)$ on the strong discontinuity is unknown too. In the case of weak discontinuity, that always propagates with the characteristic speed, motion of discontinuity against the known background is known also. In this case there are discontinuity of the second derivations or of derivations of higher order of the function $x(t,m)$, and values of this derivatives are defined by motion equations and its' differential continuations (by transport equations [5]).

2. Discontinuities of small amplitude. We will consider the class of solutions to the equations (3) with the weak discontinuity, that is created by piston's analytical moving of the form

$$x_p(t)=x_0(0)+\alpha_2t^2/2+\alpha_3t^3/6+\dots.$$

We restrict our consideration to the case of motion of discontinuity against the known static backgroud $x_0(m)$ (at $h(x)$ and $p_1(x)$) for simplicity.

In work [6] in the theory of an ideal magnetohydrodynamics were developed the method of solution of the one-dimensional problems with weak discontinuity, which moves against an arbitrary background. We will expound this method here in the more general lagrangian form. We will use the designations $v = x_\tau$ and $w = x_m(\tau, m)$.

The definition of the background speed of sound $1/T'_0$ is $\Lambda^0_{vv} = 0$. There is $\Lambda^0_{vw} > 0$ and $\Lambda^0_{vvv} < 0$ for lagrangian (4). Let $\alpha_2 \neq 0$. Then the solution of the first transport equation (Riccati's equation [5]) is defined the acceleration of the liquid on the discontinuity

$$v_\tau = \frac{1}{(\Lambda^0_{vw})^{1/2}} \left(C_2 + \int\limits_0^m \frac{\Lambda^0_{vvv}}{2(\Lambda^0_{vw})^{3/2}} dm \right)^{-1} . \tag{6}$$

Calculating of all derivatives of the background is providing under the condition $v_0 = 0$.

It should be mentioned that the value of acceleration has an order of one, but the velocity is equals to zero exactly. All the next derivatives with respect to the τ of law of motion are defined by linear equations in quadratures. The constant C_2 is proportional to the initial acceleration of the piston $\alpha_2 \neq 0$. If the $\alpha_2 < 0$, the acceleration of liquid is always negative. If the $\alpha_2 > 0$, there is a possibility for acceleration to take the infinity value in finite time (overturning of weak discontinuity). The case of $\alpha_2 = 0$ is correspond to special solution of the Riccati's equation $v_\tau = 0$.

Let us consider now the following situation: there is small, in relation to the background speed of sound, initial velocity of the piston $\alpha_1 > 0$, that is created weak shock wave. Then we linearized the second condition on the discontinuity (5) and find the correction to the value of T'_0, which correspond to the speed of shock wave,

$$\delta T' = -\Lambda^0_{vvv}[v]/(2\Lambda^0_{vvT'}).$$

Then we linearize the equation of motion in relation to the small velocity v on the discontinuity, solve the correspond linear differential equation and get using (6)

$$v = \frac{C_1}{(\Lambda^0_{vw})^{1/2}} \left| 1 + \int\limits_0^m \frac{\Lambda^0_{vvv}}{2C_2(\Lambda^0_{vw})^{3/2}} \right|^{-1/2} . \tag{7}$$

Constant C_1 could be expressed via α_1.

Formula (7) shows that if the C_2 is positive, besides the overturning of the weak discontinuity when $v_\tau \to \infty$, the velocity v for any value of $C_1 > 0$ grows unboundedly too. This says that the shock wave becomes strong and the appropriate theory is not applicable on the merits.

In the case of equillibrium the second equation of (2) gives

$$\rho g \zeta_0 + p_1 = C_0, \quad w_0 = \frac{g}{C_0 - p_1 + \rho g h}. \tag{8}$$

Calculating the derivatives of lagrangian (4) gives

$$\Lambda^0_{vw} = gT'_0/(\rho w^3), \quad \Lambda^0_{vvv} = -3g(T'_0)^3/(\rho w_0^4), \quad T'_0 = (\rho w_0^3/g)^{1/2}.$$

It is convenient to go to the variable x, taking into account that $w\,dm = dx$. Then for the acceleration of liquid on the weak discontinuity we get

$$v_\tau = (g/\rho) w_0^{3/4} \left(C_2 - \int\limits_{x_p(t)}^{x_0(t)} (3/2) w_0^{7/4} dx \right)^{-1} ,$$

and for the velocity on the weak shock wave we get

$$v = C_1(\rho/g)^{1/4} w_0^{3/4} \left| 1 - \int\limits_{x_p(t)}^{x_0(t)} (3/(2C_2)) w_0^{7/4} dx \right|^{-1/2},$$

where $x_0(t)$ corresponds to the motion of the background's characteristic $t = T_0(m)$.

Analysis of the formulae shows that singularity, which is connected with equality to zero of denominator of the specific volume w_0 (8) when the discontinuity tends to the zero thickness of the layer $\zeta_0 + h = 0$, can appear. It is possible only when the acceleration of piston is negative. In this case we can trace the attenuation process of the shock wave [6, 7]. However if the acceleration is positive before this situation the overturning of weak discontinuity occurs.

Acknowledgements. The work was provided with partial financial support of the Russian Foundation for Basic Research (No, No 15-01-00361, 17-01-00037).

REFERENCES

1. L. D. Landau L.D., E. M. Lifshitz. *Fluid Mechanics*. Pergamon Press, 1987.
2. S. Leibovich, A. R. Seebass. Examples of Dissipative and Dispersive Systems Leading to the Burgers and the Korteweg-de Vries Equations. *Collection of Articles "Nonlinear Waves"*. Cornell University Press, 1974, pp.103–138.
3. L. I. Sedov. *A Course in Continuum Mechanics*. Volumes I–IV. Wolters-Noordhoff Publishing, 1971.
4. L. D. Landau, E. M. Lifshitz. *Electrodynamics of Continuous Media*. Pergamon Press, 1984.
5. B. L. Rozdestvenskii, N. N. Yanenko. *Systems of Quasilinear Equations and Their Applications to Gas Dynamics*. Providence-Rhode Island: S.n., 1983. (American Mathematical Society. Translations of Mathematical Monographs. Volume 55)
6. A. N. Golubyatnikov, S. D. Kovalevskaya. Propagation of Discontinuities against a Static Background. *Fluid Dynamics*. Volume 52, Issue 2 (2017), pp.321–328.
7. G. G. Chernyi. *Gas Dynamics* [in Russian]. Moscow, Nauka, 1988.