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Fluid Outflow from an Orifice in a Plane Wall in the Presence of a Variable-Strength Source on the Symmetry Plane of Flow

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Abstract—The unsteady jet outflow of an ideal incompressible weightless fluid from an orifice in a wall is considered in the presence of a variable-strength point source on the symmetry plane of flow. It is assumed that the velocity of the perturbed flow induced by the change of the source discharge is small compared to the velocity of the steady flow. The Gurevich–Haskind method is used to solve the problem. A boundary value problem for the complex potential of the perturbed flow is formulated and solved. The pressure distribution on hard walls is determined for the harmonic law of the source discharge variation. The evolution of the shape of the jet's free boundary is studied.

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The problem on the steady jet outflow from a slot in a plane wall under the action of pressure difference is classical and its solution is well known [1]. Such problems are studied in a number of works on the basis of the linear theory of small perturbations. The deformation of a jet outflowing from an orifice in a plane wall into a medium where the pressure changes with time is studied in [2]. The jet outflow from an orifice of variable width is discussed in [3]. The unsteady jet outflow from channels is considered in [4, 5] when the laws of variation of velocity or pressure are given at infinity. The stability of a jet outflowing from a slot is analyzed in [6].

The problem on the steady jet outflow is solved in [7] in the presence of a source on the symmetry plane of flow. Figure 1a illustrates such a steady flow considered in [7]. The flow velocity is equal to zero at the infinitely remote point D of a vessel, whereas the pressure $P_{0\infty}$ at this point is greater than the pressure P_{0C} at the region of jet flow. Based on the Bernoulli integral, it can be concluded that, on the free surfaces of the jet, the absolute value V_{0C} of the velocity is constant and is equal to $\sqrt{2(P_{0\infty} - P_{0C})/\rho}$, where ρ is the density of the fluid. The free surfaces of the jet are denoted by EB and FB; the width of the orifice in the wall FE is denoted by H. A source of strength $Q_0 > 0$ is situated at the point C on the symmetry plane of flow. For the scheme given in Fig. 1a, the flow discharge is equal to zero at the infinitely remote point D, whereas the jet discharge is equal to the source discharge.

In this paper we consider the unsteady flow caused by a small time variation of the source discharge according to the law $Q(t) = Q_0 + q(t)$. The pressure is constant on the free surface; the walls of the channels are fixed. It is assumed that the velocity of the perturbed flow is small compared to the velocity of the steady flow.

The upper half-plane u is chosen as a parametric domain. The correspondence of the points is illustrated in Figs. 1a and 1c for the physical and parametric planes z and u.

The complex potential of the unsteady flow is the following sum of the complex potentials of the steady and perturbed flows: $w(u, t) = w_0(u) + w(u, t)$.

According to [8], below we formulate and solve a boundary value problem for the Laplace transform $\overline{w}(u, s)$ rather than for the function $w(u, t) = \varphi(u, t) + i\psi(u, t)$.

For the steady flow, we have

$$w_0(u) = \frac{Q_0}{\pi} \ln u, \qquad z = z(u). \tag{1}$$

The function $z(u)$ can be determined by solving the stationary problem discussed in [7].

Since the hard walls are fixed, the following equality is valid on them: $\dfrac{\partial\varphi}{\partial n} = 0$; hence, $\psi = \psi(t)$. It is assumed that the change of the source discharge does not lead to the discharge change at the infinitely remote point D, otherwise the pressure becomes infinite at the point D, since the complex potential $w(u, t)$

of the perturbed flow has the logarithmic singularities at the points $u = \pm d$. Thus, we can assume that $\psi = -q(t)/2$ for $u \in (0,1)$ and $\psi = q(t)/2$ for $u \in (-1,0)$.

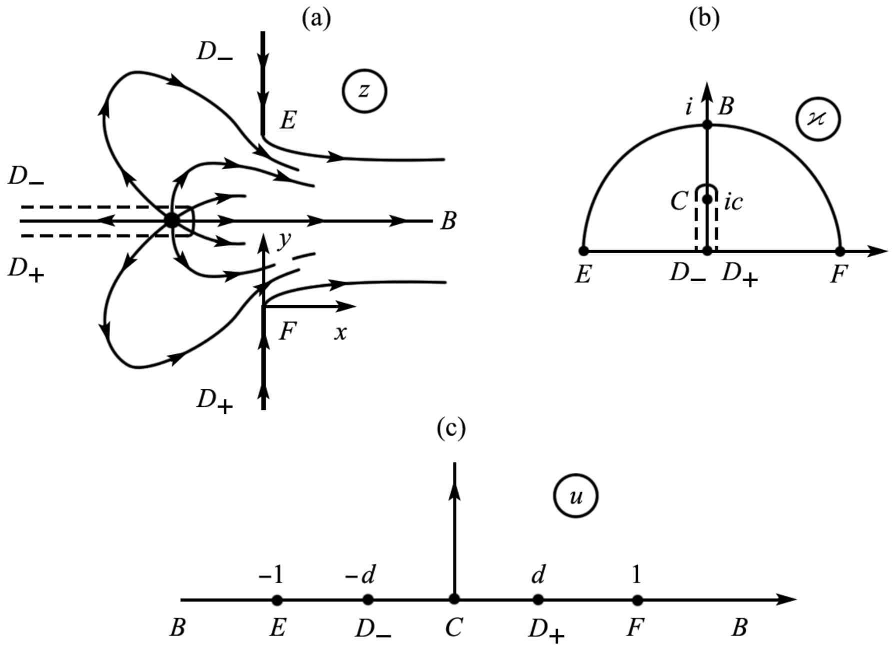

Fig. 1. (a) Scheme of flow. (b, c) Parametric domains.

The Cauchy–Lagrange integral takes the form

$$P = P_{0\infty} - \rho\left[\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\mathbf{V},\mathbf{V})\right],$$

where P is pressure, $\mathbf{V} = \mathbf{V_0} + \nabla\varphi$, and $\mathbf{V_0}$ is the velocity of the steady flow.

Using the Bernoulli integral with an accuracy up to $O(\nabla\varphi)$, we get

$$\Delta P = P - P_0 = -\rho\left[\frac{\partial\varphi}{\partial t} + (\mathbf{V}_{0C},\nabla\varphi)\right], \tag{2}$$

where P_0 is the pressure in the steady flow.

The pressure is constant in the surrounding medium; on the free surface, hence, we have $\Delta P = 0$. From (2) we come to the following relation on the free surface [9]:

$$\frac{\partial\varphi}{\partial t} + V_{0C}^2\frac{\partial\varphi}{\partial\varphi_0} = 0. \tag{3}$$

Here $\varphi_0 = \operatorname{Re} w_0(u)$ is the potential of the unperturbed flow. The simplified dynamic condition (3) has an error of order $\eta\kappa$, where κ is the curvature of the free surface and η is the normal component of the deviation of the free surface from its stationary position [9, 10].

Let $\tau = V_{0C}^2 t/Q_0$ be the dimensionless time. Then, the condition expressed by (3) takes the form

$$\frac{1}{Q_0}\frac{\partial\varphi}{\partial\tau} + \frac{\partial\varphi}{\partial\varphi_0} = 0.$$

Using the Laplace transforms, we get $\dfrac{s}{Q_0}\,\overline{\varphi} + \dfrac{d\overline{\varphi}}{d\varphi_0} = 0$; therefore, $\dfrac{1}{Q_0}\,\overline{\varphi} = D_2(s)e^{-\frac{s}{Q_0}\varphi_0}$.

From (1) it follows that $\varphi_0(u) = Q_0 \ln|u|/\pi|$ for $u \in (-\infty,-1)\cup(1,\infty)$. Thus, the function $\overline{w}(u,s)$ is analytical in the upper half-plane and satisfies the following boundary conditions on the real axis:

$$\begin{cases} \operatorname{Im} \dfrac{1}{Q_0}\overline{w}(u,s) = \begin{cases} D_1(s) = \dfrac{\overline{q}(s)}{2Q_0}, & u \in (-1,0); \\ -D_1(s), & u \in (0,1); \end{cases} \\ \operatorname{Re} \dfrac{1}{Q_0}\,\overline{w}(u,s) = D_2(s)|u|^{-s/\pi}, \qquad u \in (-\infty,-1) \cup (1,\infty). \end{cases}$$

The above mixed boundary value problem can be solved using the Keldysh–Sedov formula [11]

$$\frac{1}{Q_0}\,\overline{w}(u,s) = \frac{2\sqrt{u^2-1}}{\pi i}\left\{ D_2(s)\int_1^{\infty} \frac{\xi\xi^{-s/\pi}\,d\xi}{\sqrt{\xi^2-1}(\xi^2-u^2)} - D_1(s)\int_0^1 \frac{\xi\,d\xi}{\sqrt{1-\xi^2}(\xi^2-u^2)} \right\}. \tag{4}$$

From (4) we conclude that for $|u| < 1$ the potential can be written as

$$\frac{1}{Q_0}\,\overline{\varphi}(u,s) = \frac{2D_2(s)\sqrt{1-u^2}}{\pi}\int_1^{\infty} \frac{\xi^{-s/\pi+1}\,d\xi}{(\xi^2-u^2)\sqrt{\xi^2-1}} - \frac{D_1(s)}{\pi}\ln\frac{1+\sqrt{1-u^2}}{1+\sqrt{1-u^2}}. \tag{5}$$

From (5) we obtain

$$\frac{1}{Q_0}\,\frac{d\overline{\varphi}(u,s)}{du} = \frac{2u}{\pi\sqrt{1-u^2}}\left[\frac{sD_2(s)}{\pi}\int_1^{\infty} \frac{\xi^{-s/\pi-1}\sqrt{\xi^2-1}\,d\xi}{\xi^2-u^2} + \frac{D_1(s)}{u^2}\right]. \tag{6}$$

The velocity of the perturbed flow should be finite at the points E and F. Hence, the expression enclosed in the square brackets of (6) should be equal to zero for $u = \pm 1$. Then, we obtain

$$D_2(s) = -\frac{\pi D_1(s)}{sJ}, \qquad J = \int_1^{\infty} \frac{\xi^{-s/\pi-1}\,d\xi}{\sqrt{\xi^2-1}}.$$

The excess pressure coefficient $C_p = \dfrac{2\Delta P}{\rho V_{0C}^2}$ can be found by (2) on the wall D_+F for $u \in (d,1]$:

$$\overline{C}_p(u,s) = \frac{2\overline{\Delta P}(u,s)}{\rho V_{0C}^2} = -s\,\frac{\overline{\varphi}(u,s)}{Q_0} - \frac{V_0^2(u)}{V_{0C}^2}\,\pi u\,\frac{d\overline{\varphi}(u,s)}{Q_0\,du}. \tag{7}$$

The functions $\overline{\varphi}(u,s)$ and $\dfrac{d\overline{\varphi}(u,s)}{du}$ used in (7) are defined by (5) and (6), respectively. In order to find the distribution of V_0 on the wall D_+F, we use the results discussed in [7], where the sought solution is obtained by mapping the variation domains of the complex potential and the complex velocity of the steady flow onto the variation domain of the parametric variable $\varkappa$, i.e., onto the semicircle of unit radius (Fig. 1b). In order to find a dependence of these quantities on the variable u, we use the following conformal mapping of u onto $\varkappa$:

$$\varkappa = \frac{\sqrt{u^2-d^2}}{\sqrt{1-d^2}+\sqrt{1-u^2}}. \tag{8}$$

Here $d = \dfrac{2c}{1+c^2}$. The parameter c specifies the position of the source in the parametric domain $\varkappa$. Its value is dependent on the dimensionless source discharge $q_0 = Q_0/(V_{0C}H)$ and on its position on the symmetry plane of flow [7]. For $u \in (d,1]$ we get

$$\frac{V_0^2(u)}{V_{0C}^2} = \frac{\varkappa^6(1+c^2\varkappa^2)^2}{(\varkappa^2+c^2)^2},$$

where the dependence $\varkappa = \varkappa(u)$ is given by (8).

The domains z and $\varkappa$ are related as [7]

$$\frac{z}{H} = \frac{i(F(\varkappa) - F(1))}{F(-1) - F(1)}, \qquad F(\varkappa) = \int \frac{(\varkappa^2 - 1)\, d\varkappa}{\varkappa^2(\varkappa^2 + 1)(1 + c^2\varkappa^2)^2}.$$

The expression of the function $F(\varkappa)$ is cumbersome and is not given in this paper.

The following kinematic condition can be used to determine the deformations of the free surfaces [10]:

$$\frac{\partial \eta}{\partial t} + V_{0C}^2 \frac{\partial \eta}{\partial \varphi_0} = -V_{0C} \frac{\partial \psi}{\partial \varphi_0}.$$

Using the dimensionless time τ and the Laplace transform, we rewrite this condition as

$$\frac{s}{H}\, \overline{\eta} + \frac{Q_0}{H} \frac{d\overline{\eta}}{d\varphi_0} = -q_0 \frac{d\overline{\psi}}{d\varphi_0}. \tag{9}$$

In order to solve the differential equation (9), we should obtain an explicit form of its right-hand side. Using (4), for $|u| > 1$ we get

$$\frac{1}{Q_0}\, \overline{\psi}(u, s) = -\frac{2\sqrt{u^2 - 1}}{\pi} D_2(s) \int_1^{\infty} \frac{\xi \xi^{-s/\pi}\, d\xi}{\sqrt{\xi^2 - 1}(\xi^2 - u^2)} + \frac{2D_1(s)}{\pi} \arccos \frac{\sqrt{u^2 - 1}}{u}.$$

Based on the change of variables

$$\frac{1}{u} = \cos\frac{\alpha}{2}, \qquad \frac{1}{\xi} = \cos\frac{\theta}{2}, \qquad \alpha, \theta \in [0, \pi),$$

now we evaluate the integral $I = \int\limits_1^{\infty} \frac{\xi \xi^{-s/\pi}\, d\xi}{\sqrt{\xi^2 - 1}(\xi^2 - u^2)}$. Since

$$\left(\cos\frac{\theta}{2}\right)^{s/\pi} = \sum_{k=0}^{\infty} a_k(s) \cos k\theta, \qquad \int_0^{\pi} \frac{\cos k\theta}{\cos\alpha - \cos\theta} = -\pi \frac{\sin k\alpha}{\sin\alpha},$$

we get $I = -\frac{\pi \cos^2 \frac{\alpha}{2}}{\sin\alpha} \sum\limits_{k=0}^{\infty} a_k(s) \sin k\alpha$. Hence, the equation expressed by (9) can be rewritten as

$$\frac{s}{2\pi} \sin\frac{\alpha}{2} \frac{\overline{\eta}(\alpha, s)}{H} + \cos\frac{\alpha}{2} \frac{d\overline{\eta}(\alpha, s)}{H d\alpha} = -q_0 \cos\frac{\alpha}{2} \left(D_2(s) \sum_{k=1}^{\infty} k a_k(s) \cos k\alpha + \frac{D_1(s)}{\pi} \right). \tag{10}$$

Equation (10) is solved with the boundary condition $\overline{\eta}(0, s) = 0$.

Our numerical results were obtained for the following harmonic law of source discharge variation: $q(t) = \varepsilon e^{jkt}$ or $q(\tau) = \varepsilon e^{2j\omega\tau}$, where $\omega = \frac{kQ_0}{2V_{0C}^2}$ is the angular frequency and j is the imaginary unit.

It is assumed that the oscillations continue for a long time. Therefore, we can ignore the effect of the transient process caused by the oscillations appeared at the time instant $\tau = 0$. We can also assume that the complex potential of the perturbed flow is a time-periodic function with the same angular frequency as for the function $q(\tau)$. In this case it is not necessary to find the inverse transforms, since the sought solution is specified by the Laplace transforms for $s = 2j\omega$ [10].

Our numerical results show that the harmonic time variation of the source discharge leads to the appearance of a wave on the free surfaces of the jet. As the distance from the orifice increases, this wave can be considered as a traveling wave moving with the velocity V_{0C} near the stationary free surface of almost zero curvature. The amplitude of this wave is proportional to the amplitude of the perturbing action ε; the length of this wave decreases with increasing ω. Figure 2 illustrates the shape of the free surface FB for various time instances and for $\varepsilon = 0.1$, $\omega = 1$, and $q_0 = 0.7$. The source is situated at the point $Z_C = -0.54H + 0.5Hi$.

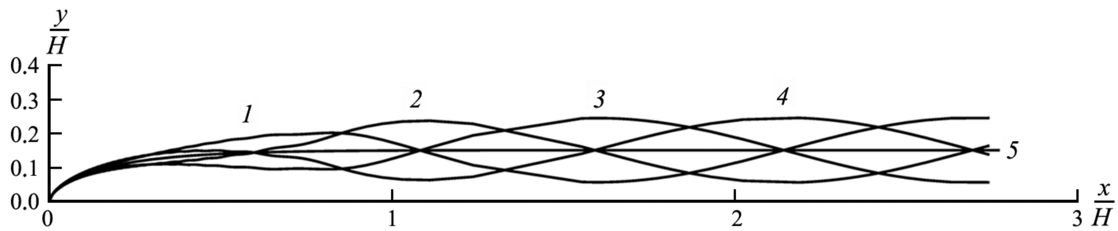

Fig. 2. The shape of the free boundary FB for $\tau = 3\pi/4$, 0, $\pi/4$, $\pi/2$ (curves *1–4*, respectively) and for $\omega = 1$, $\varepsilon = 0.1$, $q_0 = 0.7$, and $Z_C = -0.54H + 0.5Hi$; (*5*) the stationary free surface.

Figure 3 illustrates the distribution of the amplitude for the pressure coefficient $\lambda(\omega, y) = |\overline{\Delta C_p}|$ on the wall D_+F for $\varepsilon = 0.1$ and $q_0 = 0.7$ and for several values of ω. As the distance from the orifice increases, the value of λ monotonically increases from zero and asymptotically approaches a value dependent on ω. From Fig. 3 it follows that λ increases with increasing ω.

In conclusion we note that the process of solving the nonstationary problems for other possible modes of steady flow with a nonzero discharge at the infinitely remote point D can be performed according to the above discussion. The only distinctions are the form of the function $\varphi_0(u)$ in the mixed boundary value problem for the complex potential of the perturbed flow and the forms of the formulas used for the complex potential and for the complex velocity of the corresponding steady flow.

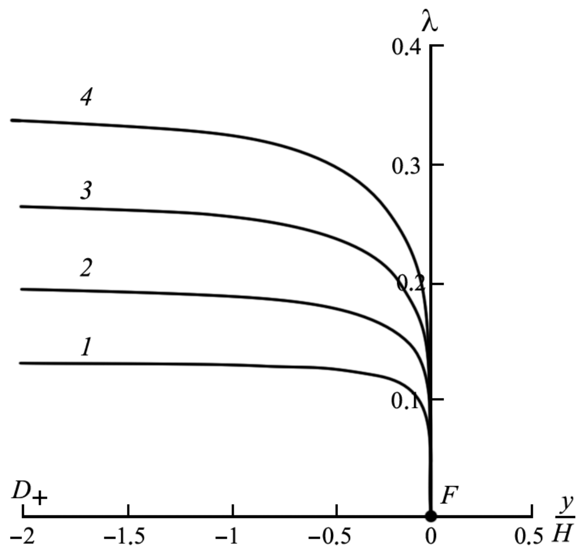

Fig. 3. The distribution of the amplitude of the pressure coefficient on the wall D_+F for $\omega = 1, 2, 3, 4$ (curves *1–4*, respectively) and for $\varepsilon = 0.1$, $q_0 = 0.7$, and $Z_C = -0.54H + 0.5Hi$.

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