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Fundamental Solution of Displacement Equations for a Transversely Isotropic Elastic Medium

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Abstract—A fourth-order linear elliptic partial differential equation describing the displacements of a transversely isotropic linear elastic medium is considered. Its symmetries and the symmetries of an inhomogeneous equation with a delta function on the right-hand side are found. The latter symmetries are used to construct an invariant fundamental solution of the original equation in terms of elementary functions.

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INTRODUCTION AND THE MAIN RESULT

In [1], the system of displacement equilibrium equations for a transversely isotropic linear elastic medium is reduced to a system of three linear inhomogeneous equations for three displacement components. The homogeneous equations are associated with canonical linear partial differential equations of the fourth order. These canonical equations are a generalization of the biharmonic equation describing the displacements of an isotropic linear elastic medium. To find the displacements of a transversely isotropic linear elastic medium subjected to a given body force, we need to know fundamental solutions of the canonical equations.

Note that reductions of the system of displacement equations in 3D elasticity to systems of higher order equations based on operators that are more suitable for a numerical-analytical study than the Lamé operator are called representations of the solution of the elasticity problem and are described in the classical theory. Specifically, a reduction to tetraharmonic equations was reduced in [2].

Fundamental solutions of linear partial differential equations are frequently invariant under transformations admitted by the original equation [3]. Below, a fundamental solution is constructed using the algorithm from [4] proposed for finding fundamental solutions of linear partial differential equations. The algorithm makes use of the symmetries admitted by a linear partial differential equation with a delta function on its right-hand side. Let us briefly describe the main result of this work.

Consider the pth-order linear partial differential equation

$$Lu \equiv \sum_{\alpha=1}^{p} A_\alpha(x) D^\alpha u = 0, \qquad x \in R^m. \tag{1}$$

Here, the standard notation is used: $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multi-index with nonnegative integer components, $\alpha = \alpha_1 + \cdots + \alpha_m$, and

$$D^\alpha \equiv \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^m}\right)^{\alpha_m}.$$

The fundamental solutions of Eq. (1) are solutions of the equation

$$Lu = \delta(x - x_0). \tag{2}$$

It was shown in [5] that Eq. (1) with $p \geq 2$ and $m \geq 2$ can admit only symmetry operators of the form

$$Z = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} + \eta(x,u) \frac{\partial}{\partial u}, \qquad \frac{\partial^2 \eta}{\partial u^2} = 0.$$

The basic Lie algebra of symmetry operators of Eq. (1) regarded as a vector space is the direct sum of two subalgebras: one consisting of operators of the form

$$X = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} + \zeta(x) u \frac{\partial}{\partial u} \tag{3}$$

and the infinite-dimensional subalgebra generated by the operators

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$$X_\infty = \varphi(x)\frac{\partial}{\partial u}, \tag{4}$$

where $\varphi(x)$ is an arbitrary solution of Eq. (1). Note that operators (4) are symmetry operators of Eq. (2). In what follows, we consider only symmetry operators of form (3). Let $\underset{p}{X}$ denote an extension of order p of symmetry operator (3).

Proposition 1. *The infinitesimal operator given by* (3) *is a symmetry operator of Eq.* (1) *if and only if there exists a function* $\lambda = \lambda(x)$ *satisfying the identity*

$$\underset{p}{X}(Lu) \equiv \lambda(x)Lu \tag{5}$$

for any function $u = u(x)$ *from the domain of Eq.* (1).

Theorem 1. *The Lie algebra of symmetry operators of Eq.* (2) *is a subalgebra of the Lie algebra of symmetry operators of Eq.* (1) *and is defined by the relations*

$$\begin{gathered}\xi^i(x_0) = 0, \quad i = 1, 2, \dots, m,\\ \lambda(x_0) + \sum_{i=1}^{m}\frac{\partial \xi^i(x_0)}{\partial x^i} = 0.\end{gathered} \tag{6}$$

Let us describe an algorithm for finding fundamental solutions by applying symmetries [4]:

1. Find a general symmetry operator of Eq. (1) and the corresponding function $\lambda(x)$ satisfying identity (5).

2. Use this operator and relations (6) to obtain the basis for the Lie algebra of symmetry operators of Eq. (2).

3. Construct invariant fundamental solutions with the help of the symmetries of Eq. (2).

4. Obtain new fundamental solutions from the known ones with the help of the symmetries of Eq. (2) (production of solutions).

Remark 1. To find generalized invariant fundamental solutions, we need to search for invariants in the class of generalized functions.

The main result of this paper is the construction (in terms of elementary functions) of an invariant fundamental solution to the equation of a transversely isotropic linear elastic medium.

THE BASIC EQUATIONS

Consider the following fourth-order linear differential equations, which were introduced in [1]:

$$\begin{aligned}L_1 u &\equiv u_{xxxx} + 2u_{xxyy} + u_{yyyy}\\ &+ B_1(u_{xxzz} + u_{yyzz}) + B_2 u_{zzzz} = 0,\\ L_2 u &\equiv B_3(u_{xxxx} + 2u_{xxyy} + u_{yyyy})\\ &+ B_4(u_{xxzz} + u_{yyzz}) + u_{zzzz} = 0.\end{aligned} \tag{7}$$

Here, B_1, B_2, B_3, and B_4 are positive constants characterizing a linear elastic medium. The fundamental solutions of Eqs. (7) are solutions of the equations

$$L_1 u = \delta(x)\delta(y)\delta(z), \qquad L_2 u = \delta(x)\delta(y)\delta(z). \tag{8}$$

Let us show that Eqs. (7) and (8) can be reduced to identical equations by changing variables. For this purpose, in the equations for the differential operator L_1, we pass to the new variables

$$\bar{z} = \frac{z}{\sqrt[4]{B_2}}, \qquad \bar{u} = \sqrt[4]{B_2}\,u.$$

After omitting the bars over the new variables, the corresponding equations (7) and (8) become

$$\begin{aligned}L_3 u &\equiv u_{xxxx} + 2u_{xxyy} + u_{yyyy}\\ &+ b(u_{xxzz} + u_{yyzz}) + u_{zzzz} = 0,\end{aligned} \tag{9}$$

$$L_3 u = \delta(x)\delta(y)\delta(z). \tag{10}$$

Here, $b = \dfrac{B_1}{\sqrt{B_2}}$. Similarly, by changing to the variables

$$\bar{x} = \frac{x}{\sqrt[4]{B_3}}, \qquad \bar{y} = \frac{y}{\sqrt[4]{B_3}}, \qquad \bar{u} = \sqrt{B_3}\,u,$$

the equations for the differential operator L_2 are reduced to Eqs. (9) and (10) with $b = \dfrac{B_4}{\sqrt{B_3}}$.

Assume that Eq. (9) is elliptic. Then it must hold that $b \ge 2$.

In what follows, Eq. (9) is considered the basic equation.

The axisymmetric solutions of Eq. (9) satisfy the equation

$$\begin{aligned}L_4 u &\equiv u_{rrrr} + bu_{rrzz} + u_{zzzz} + \frac{2}{r}u_{rrr}\\ &+ \frac{b}{r}u_{rzz} - \frac{1}{r^2}u_{rr} + \frac{1}{r^3}u_r = 0,\end{aligned} \tag{11}$$

while the axisymmetric fundamental solutions (or axisymmetric solutions of Eq. (10)) satisfy the equation

$$rL_4 u = \frac{1}{2\pi}\delta(r)\delta(z). \tag{12}$$

Here, $r = \sqrt{x^2 + y^2}$ and

$$\int_0^\infty \delta(r)dr = 1.$$

Equation (12) can be rewritten in conservative form as

$$\begin{aligned}&\left(ru_{rrr} + bru_{rzz} + u_{rr} - \frac{1}{r}u_r\right)_r + (ru_{zzz})_z\\ &= \frac{1}{2\pi}\delta(r)\delta(z).\end{aligned} \tag{13}$$

SYMMETRIES OF THE BASIC EQUATIONS

The symmetries of Eq. (9) can be found using the symmetry-finding algorithm from [3].

Proposition 2. *Equation* (9) *with an arbitrary parameter b admits the following basis of the Lie algebra of symmetry operators:*

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = \frac{\partial}{\partial z},$$
$$X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$X_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$
$$X_6 = u\frac{\partial}{\partial u}, \quad X_\infty = \varphi(x, y, z)\frac{\partial}{\partial u}.$$

For $b = 2$, the basis of the Lie algebra is supplemented with the symmetry operators

$$X_7 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z},$$
$$X_8 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$
$$X_9 = (x^2 - y^2 - z^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + xu\frac{\partial}{\partial u},$$
$$X_{10} = 2xy\frac{\partial}{\partial x} + (y^2 - x^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} + yu\frac{\partial}{\partial u},$$
$$X_{11} = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 - x^2 - y^2)\frac{\partial}{\partial z} + zu\frac{\partial}{\partial u}.$$

Here, $u = \varphi(x, y, z)$ is an arbitrary solution of Eq. (9).

To find the symmetries of Eq. (10), we use the results of [4]. Using the finite-dimensional part of the Lie algebra of symmetry operators of Eq. (9), we consider the general symmetry operator

$$X = \sum_{i=1}^{6} a_i X_i.$$

Here, a_i $(i = 1, 2, \ldots, 6)$ are arbitrary constants.

Proposition 3. *It is true that*

$$\underset{4}{X}\, L_3u = (a_6 - 4a_5)L_3u.$$

Then, using Theorem 1, we find that

$$a_1 = a_2 = a_3 = 0, \quad a_5 - a_6 = 0,$$

which yield the following result.

Proposition 4. *For an arbitrary parameter b, Eq.* (10) *admits the following basis of the Lie algebra of symmetry operators*:

$$Y_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$Y_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}. \tag{14}$$

Remark 2. It can also be shown that, for $b = 2$, Eq. (10) admits symmetry operators (14), the symmetry operator

$$Y_3 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \tag{15}$$

and the symmetry operators X_8, X_9, X_{10}, and X_{11}.

FUNDAMENTAL SOLUTION

Let us find a solution of Eq. (9) that is invariant under symmetry operators (14). The invariants of the admitted transformation group are $J_1 = \frac{r^2}{z^2} = \tau$ and $J_2 = \frac{u}{z}$. Then an invariant solution is sought in the form

$$u = zf(\tau). \tag{16}$$

Substituting (16) into Eq. (9) (or Eq. (11)), we obtain the fourth-order ordinary differential equation

$$\begin{aligned} &4\tau^2(\tau^2 + b\tau + 1)\frac{d^4 f}{d\tau^4} \\ &+ 2\tau(14\tau^2 + 11b\tau + 8)\frac{d^3 f}{d\tau^3} \\ &+ (39\tau^2 + 22b\tau + 8)\frac{d^2 f}{d\tau^2} + 2(3\tau + b)\frac{df}{d\tau} = 0. \end{aligned} \tag{17}$$

Proposition 5. *The ordinary differential equation* (17) *has the following fundamental set of solutions*:

$$f_1 = 1,$$
$$f_2 = \sqrt{\frac{\tau}{a} + 1} - \operatorname{arcoth}\sqrt{\frac{\tau}{a} + 1} + \sqrt{a\tau + 1} - \operatorname{arcoth}\sqrt{a\tau + 1},$$
$$f_3 = \frac{a}{a^2 - 1}\left(\sqrt{\frac{\tau}{a} + 1} - \operatorname{arcoth}\sqrt{\frac{\tau}{a} + 1} - \sqrt{a\tau + 1} + \operatorname{arcoth}\sqrt{a\tau + 1}\right),$$
$$f_4 = \frac{a}{a^2 - 1}\left(\sqrt{\frac{\tau}{a} + 1}\cdot\operatorname{arcoth}\sqrt{\frac{\tau}{a} + 1} - \frac{1}{2}\operatorname{arcoth}^2\sqrt{\frac{\tau}{a} + 1} - \sqrt{a\tau + 1}\cdot\operatorname{arcoth}\sqrt{a\tau + 1} + \frac{1}{2}\operatorname{arcoth}^2\sqrt{a\tau + 1}\right),$$

where the parameter a satisfies the relation $b = a + \frac{1}{a}$.

Consider the general solution of Eq. (17):

$$f = \sum_{i=1}^{4} c_i f_i, \tag{18}$$

where c_i $(i = 1, 2, \ldots, 4)$ are arbitrary constants. Among solutions (18), we find ones that take finite values, together with their first derivatives, at $\tau = 0$.

Proposition 6. *As* $\tau \to 0$, *we obtain*

$$f = \left[c_2 + \frac{a\ln a}{2(a^2-1)}c_4\right]\ln\tau + O(1),$$

$$\frac{df}{d\tau} = \left[c_2 + \frac{a\ln a}{2(a^2-1)}c_4\right]\frac{1}{\tau} + \frac{c_4}{8}\ln\tau + O(1).$$

It follows that $c_2 = 0$ and $c_4 = 0$. Assume also that $c_1 = 0$. As a result, we obtain the following one-parameter family of solutions to Eq. (17):

$$f = \frac{c_3 a}{a^2-1}\left(\sqrt{\frac{\tau}{a}+1} - \sqrt{a\tau+1} - \operatorname{arcoth}\sqrt{\frac{\tau}{a}+1} + \operatorname{arcoth}\sqrt{a\tau+1}\right).$$

Then, using (16) yields a one-parameter family of solutions to Eq. (9) (or to Eq. (11)):

$$u = \frac{c_3 a}{a^2-1}\left(\sqrt{\frac{r^2}{a}+z^2} - \sqrt{ar^2+z^2} - z\cdot\operatorname{arcoth}\frac{\sqrt{\frac{r^2}{a}+z^2}}{z} + z\cdot\operatorname{arcoth}\frac{\sqrt{ar^2+z^2}}{z}\right). \tag{19}$$

Let us show that solutions (19) contain a fundamental one. For this purpose, both sides of Eq. (13) are integrated over the rectangular domain $\Pi = \{0 \le r \le r_0, -z_1 \le z \le z_2, r_0 > 0, z_1 > 0, z_2 > 0\}$. By using the Stokes formula, the integral on the left-hand side can be written in terms of an integral along the boundary of Π. Then solution (19) is substituted into the resulting integrand. Finally, we find that $c_3 = \frac{1}{4\pi}$.

Below is the main result of this work.

Theorem 2. *The fundamental solution of Eq.* (9) *can be written as*

$$u_f = \frac{a}{4\pi(a^2-1)}\left[\sqrt{\frac{r^2}{a}+z^2} - \sqrt{ar^2+z^2} + \frac{z}{2}\ln\frac{\left(\sqrt{\frac{r^2}{a}+z^2}-z\right)\left(\sqrt{ar^2+z^2}+z\right)}{\left(\sqrt{\frac{r^2}{a}+z^2}+z\right)\left(\sqrt{ar^2+z^2}-z\right)}\right]. \tag{20}$$

Remark 3. For $a = 1$ (or $b = 2$), the fundamental solution (20) becomes

$$u_f = -\frac{1}{8\pi}\sqrt{r^2+z^2}$$

and coincides with the fundamental solution of the biharmonic equation [6].

Remark 4. When $b = 2$, the construction of a fundamental solution based on symmetries is especially effective. Specifically, the solution of Eq. (9) invariant under symmetry operators (14) and (15) is immediately determined up to a multiplicative constant and is given by

$$u = c\sqrt{r^2+z^2}. \tag{21}$$

Proceeding as described above, we find that (21) is a fundamental solution with $c = -\frac{1}{8\pi}$.

CONCLUSIONS

The main result of this work is the construction (in terms of elementary functions) of an invariant fundamental solution to the equation of a transversely isotropic linear elastic medium. To conclude, we note that the approach applied, which is based on using symmetries, can also be effectively used to construct fundamental solutions of linear partial differential equations with variable coefficients and for high-order equations.

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REFERENCES

1. D. V. Georgievskii, J. Appl. Math. Mech. **79** (6), 618–621 (2015).
2. D. V. Georgievskii, Dokl. Phys. **60** (8), 364–367 (2015).
3. L. V. Ovsiannikov, *Group Analysis of Differential Equations* (Nauka, Moscow, 1978; Academic, New York, 1982).
4. A. V. Aksenov, Dokl. Math. **51** (3), 329–331 (1995).
5. G. Bluman J. Math. Anal. Appl. **145** (1), 52–62 (1990).
6. V. S. Vladimirov, *Generalized Functions in Mathematical Physics* (Mir, Moscow, 1979; Nauka, Moscow, 1979).

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