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The formation of an anisotropic elastic medium on the compaction front of a stream of particles☆

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ABSTRACT

A plane transformation front of a stream of non-interacting particles into a continuous medium, which is assumed to be incompressible, elastic, anisotropic and non-linear, is considered. The structure of the compaction front is investigated using the Kelvin–Voigt model of a viscoelastic medium. It is shown that additional boundary conditions, which follow from the requirement for the existence of a discontinuity structure, must be formulated on the compaction front in certain cases along with the boundary conditions which follow from the conservation laws. These additional conditions depend on the equations that are adopted to describe the structure, and their number depends on the relations between the velocity of the front and the velocities of the small perturbations behind the front. It is shown that in the phase space of the shear strains and normal velocity of the front, the set of states behind all the possible compaction fronts (an analogue of the shock adiabat) can consist of manifolds of various dimensions (from one to three). The piston problem, which, as is shown, has a unique solution in the entire permissible region of values of shear and normal stresses assigned on the piston, is investigated.

A stream of non-interacting particles incident to a plane boundary, where the stream of particles is compacted and the particles adhere and ultimately transform into a continuous medium, which is assumed to be elastic, incompressible, weakly anisotropic and weakly non-linear, is considered in this paper. It is assumed that viscosity effects appear and that the flow is described by the Kelvin–Voigt equations in a narrow layer immediately after adhesion at large velocity gradients. Variation of the shear components of the stresses and the velocity occurs in the layer.

When flow is considered on a large scale, this layer can be treated as a discontinuity surface, and the flow in it can be treated as the structure of the discontinuity. The discontinuity is a phase transformation surface, on the different sides of which different systems of equations hold. On one side, the equations of motion of free particles hold, and on the other side the equations of elasticity theory hold. We will call this discontinuity, which has an internal structure, a compaction front. The purpose of this study is to investigate the variation of quantities in a compaction front.

A similar problem with the formation of an isotropic elastic medium was previously considered.[1] Complication of the problem by including the anisotropy of the medium formed in the treatment, apart from its methodical interest, may be caused by the study of problems of the creation of media by spraying,[2] if it is required that the medium formed would have specified anisotropy. In this case, before the particles adhere, they should also have anisotropy and should be oriented in a definite manner. The latter can be achieved, for example, by applying an electric or magnetic field. In addition, anisotropy of the medium formed can appear naturally when the particles adhere, if the free particles have a non-zero tangential velocity relative to the medium formed on the compaction front at the time of adhesion.

The problem under consideration is one of a set of problems associated with discontinuities on which a phase transformation of the medium passing through them occurs. Discontinuities of this type are often called “multi-parameter”,[1] i.e., their shock adiabat has a

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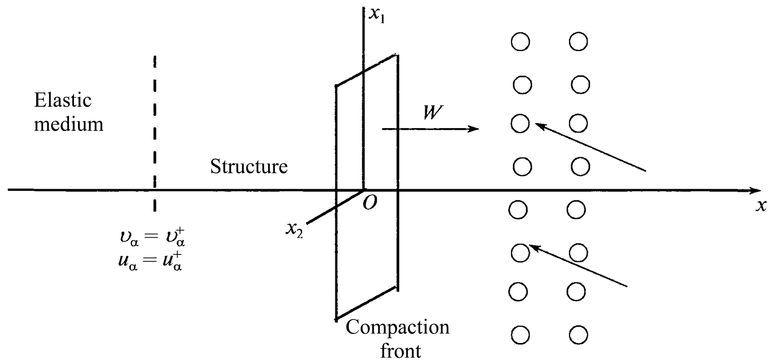

Fig. 1.

dimensionality greater than 1. One of the first examples of such multi-parameter discontinuities studied included ionization and recombination fronts in a magnetic field.[3]

There are problems which are very closely related according to the nature of the processes occurring to the problem studied below. The process of the freezing of ice from a stream of water on a stationary surface and the process of the polymerization of a stream of a medium are possible examples.

1. Statement of the problem

A homogeneous stream of non-interacting particles impinges on a plane boundary at a certain angle to it. We take the direction orthogonal to the boundary and, therefore, to the compaction front formed as the $x_3 = x$ axis of a Cartesian coordinate system (Fig. 1). The x_1 and x_2 axes lie in the plane of the front. It is assumed that all tthe flow parameters depend only on the x coordinate and the time t, i.e., the flow is assumed to be one-dimensional in the form of plane waves.

In the region ahead of the front, the velocity of the particles in the stream of free particles does not vary with time when there are no external forces, i.e., the perturbations of the motion are transported together with the particles. The velocity and density of the stream of particles are assumed to be constant and known.

As a result of passage through the compaction front, a continuous medium, which is assumed to be elastic and at rest, forms. The front moves relative to the medium formed behind it in the positive direction of the x axis with velocity W. In the system of coordinates connected to the front the components of the velocity of the stream ahead of it are

$$\upsilon_i^- = \{V_1, V_2, V_x = -V\}, \quad V > 0, \quad i = 1, 2, 3$$

and behind the front their values are $\{\upsilon_1^+, \upsilon_2^+, \upsilon_3^+\}$.

The mass conservation law on the phase transformation front has the form

$$W = V\rho_0/\rho \tag{1.1}$$

where ρ_0 and ρ are the densities in the stream of particles and in the elastic medium. When the parameters of the incident stream are given, equality (1.1) is used to determine the velocity of the front W relative to the medium formed behind it. The medium formed is assumed to be homogeneous and incompressible ($\rho = \text{const}$), and the parameters of the incident stream are assumed to be time-independent; therefore, in the entire region behind the front

$$\upsilon_x{}^+ = -W = \text{const}$$

Thus, the motion of the medium along the x axis has been determined, and only the tangential components of the velocity (υ_α), the stresses (σ_α) and the strains (u_α) (everywhere below $\alpha = 1, 2$) in the elastic medium formed are subject to further study. The quantity V and, therefore, W can take different values in different problems and are among the principal physical parameters on which the form of the solution depends.

Since an elastic medium is assumed to be incompressible, for brevity, below we will set $\rho = \text{const} = 1$.

2. Motion of the medium behind the compaction front

The elastic medium formed behind the compaction front is assumed to be incompressible, weakly non-linear and weakly anisotropic. Its velocity in Lagrangian coordinates in the direction of the x_α axis is defined in terms of the displacement vector component w_α as $\upsilon_\alpha = \partial w_\alpha/\partial t$. The elastic properties of the medium are given by its elastic potential Φ in the form of a function of the components of the shear strain $u_\alpha = \partial w_\alpha/\partial x$. Assuming that $u_\alpha \ll 1$, we can represent the elastic potential of such a medium in the form of the first principal terms of the expansion in the components u_1 and u_2, which take into account the non-linearity and anisotropy, in the form[5]

$$\Phi(u_1, u_2) = \frac{f}{2}(u_1^2 + u_2^2) + \frac{g}{2}(u_2^2 - u_1^2) - \frac{\kappa}{4}(u_1^2 + u_2^2)^2 \tag{2.1}$$

The constants f, g and κ specify the elastic properties of the medium. The first term in this expansion is the elastic potential of a linear isotropic medium. The coefficient f, i.e., the shear modulus, determines the velocity of linear transverse waves in this medium $c^o = \sqrt{f}$, since it is assumed that $\rho = 1$.

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The directions of the x_1 and x_2 axes in the plane of the front were chosen so that expansion (2.1) would not contain a term proportional to u_1u_2. Then the anisotropy of the medium in the plane of the front is characterized by a single coefficient g, which is assumed to be small, its value being such that the non-linear and anisotropic terms in expansion (2.1) are of the same small order of magnitude, i.e., $g/f \sim u_a^2$. The numbering of the u_1 and u_2 axes is assumed below to be such that $g/\kappa > 0$.

The last term in expansion (2.1) represents the influence of the small non-linearity, which is assumed to be isotropic. The non-linearity coefficient κ can have either sign, which indicates the direction of the convexity of the graph of the function $\sigma(u)$ when $g = 0$: when $\kappa > 0$, the convexity of the graph is turned upward (as is characteristic of metals), and when $\kappa < 0$, it is turned downward (as is typical of rubber and elastomers).

Expression (2.1) does not take into account the dependence of the elastic potential on the thermodynamic parameters, for example, the entropy. This can be done, because it is assumed that its value is nearly constant in the phenomena considered.

A medium with $\kappa > 0$ is considered below, because the most interesting qualitative features of the fronts studied are displayed in this case, as well as because from the analytical point of view the case when $\kappa < 0$ differs significantly from the case when $\kappa > 0$, and the simultaneous treatment of these cases would require much space.

Elastic potential (2.1) establishes the relationship between the stresses and strains

$$\sigma_\alpha = \frac{\partial \Phi}{\partial u_\alpha}, \quad u_a = u_\alpha(\sigma_1, \sigma_2) \tag{2.2}$$

The equations of the transverse motion of an elastic medium in the Lagrangian system of coordinates have the form

$$\frac{\partial \upsilon_\alpha}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial u_\alpha}\right), \quad \frac{\partial u_\alpha}{\partial t} = \frac{\partial \upsilon_\alpha}{\partial x} \tag{2.3}$$

The potential Φ is defined by expression (2.1).

System of equations (2.3) is hyperbolic and has two families of characteristics and, accordingly, two types of small perturbations, whose velocities are different due to the anisotropy. We will call them slow and fast in accordance with the values of their velocities.

In the system of coordinates connected to the medium these velocities are functions of the strain components[5,6]

$$c_{1,2}^{+} = \pm\left\{f - 2\kappa(u_1^2 + u_2^2) \mp |\kappa|\sqrt{(u_2^2 - u_1^2 - g/\kappa)^2 + 4u_1^2u_2^2}\right\}^{1/2} \tag{2.4}$$

On an unperturbed background with no strains, when $u_1 = u_2 = 0$, the velocities of the two slow waves $\pm c_1^o$, which move in opposite directions, and the velocities of the two fast waves $\pm c_2^o$ are defined by the expressions

$$c_{1,2}^{o} = \sqrt{f \mp g} \tag{2.5}$$

3. Relations on the compaction front following from the conservation laws

The mass conservation law (1.1) for $V = \text{const}$ and the medium is incompressible, as was stated above, enables us to eliminate the longitudinal motions (along the x axis) of the medium from the discussion only and to consider the motion only in planes parallel to the front.

The law of conservation of momentum for the tangential components when $\rho = 1$ gives

$$\sigma_\alpha^+ - \sigma_\alpha^- = -W(\upsilon_\alpha^+ - \upsilon_\alpha^-)$$

where σ_α^- and υ_α^- are the stresses and velocities ahead of the front, about which it is known that $\upsilon_\alpha^- = V_\alpha$ and $\sigma_\alpha^- = 0$ due to the absence of interactions in the particle stream. Thus, behind the compaction front the shear stresses in the elastic medium are defined by the expression

$$\sigma_\alpha^+ = W(V_\alpha - \upsilon_\alpha^+) \tag{3.1}$$

Since, according to the assumption made, the elastic properties of the medium do not depend on the thermodynamic parameters, a relation on the discontinuity that expresses the law of conservation of energy is not required. For transverse motions, only relations (3.1), which should be considered at all possible values of W, are of significance.

Since according to equalities (2.2), the stresses in an elastic medium are expressed in terms of the shear strain u_α^+, we can conduct the further investigation in the variables of the shear strains u_α.

Since relations (3.1), which serve as the boundary conditions, are necessary for proceeding in all cases, we will call them fundamental. We will clarify the question of the completeness of the system of boundary conditions.

4. Evolutionarity of a compaction front

For the existence of a compaction front to be possible, the condition for its evolutionarity must be satisfied. In the mathematical literature it is called the correctness condition, and it ensures a unique solution of the problem of the interaction of a discontinuity with small perturbations which are orientated in the same direction as the discontinuity. The evolutionarity condition states[7,8] that just as many boundary conditions should be set on the front as there are small perturbations going out from it plus usually one more condition, which is used to define a perturbation of the velocity of the front. In the case under consideration the velocity W of the discontinuity is determined independently, so that the evolutionarity condition requires equality between the number of relations on the discontinuity and the number of outgoing small perturbations of different types.

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Since the particles in the incident stream do not interact with each other, no small perturbations go out from the discontinuity in the upstream direction, and there are outgoing small perturbations only in the elastic medium behind the front.

Two of the transverse waves of small perturbations moving in the negative direction along the x axis are always outgoing. Two other waves moving in the same direction as the front can overtake it or lag behind, and in the latter case they also become outgoing. Thus, depending on the relationship between the velocity W of the front and the velocities of the small perturbations on the front, a different number of boundary conditions must be set.

If $W < c_1(u_1, u_2)$, only two waves moving relative to the medium in the negative direction along the x axis will be outgoing. Two boundary conditions that are necessary for evolutionarity are represented by two relations of type (3.1) that correspond to the law of conservation of transverse momentum.

When $c_1(u_1, u_2) < W < c_2(u_1, u_2)$, a slow wave moving in the positive direction lags behind the front and becomes the third outgoing wave. In this case one more additional condition besides the conservation laws indicated is required for evolutionarity on the front.

When $W > c_2(u_1, u_2)$, all four small perturbations are outgoing, and two additional conditions are necessary.

The source for obtaining the specific form of the additional boundary conditions on a discontinuity front is usually an investigation of its structure. When the general conditions that hold in the case under consideration are sufficient, the requirement for the existence of the structure at once produces the number of additional boundary conditions on the discontinuity that is specified by the conditions for its evolutionarity.[8]

5. Structure of a compaction front

The structure of a discontinuity (a compaction front in the present case) refers to a continuous solution of the equations of motion in the narrow transitional layer (which represents a discontinuity in a small-scale treatment of it), which describes the continuous variation of the flow parameters from the values ahead of the front to the values behind the front. The solution in this region is described by differential equations supplemented by dissipative terms and thus by a model that is more general than an elastic medium.

In this paper the Kelvin–Voigt model of a viscoelastic medium was adopted to describe the structure. In this model the elastic part of the stresses is defined, as before, by elastic potential (2.1) in the form $\sigma_\alpha = \partial\Phi/\partial u_\alpha$. The viscous components, which are represented by the quantities $\mu\partial\upsilon_\alpha/\partial x$, are added to them. For simplicity, the viscosity coefficient μ will be assumed to be constant. It was assumed above that $\rho = 1$; therefore, in the equations below the coefficient μ represents the kinematic viscosity. The viscous stresses become small and can be neglected for sufficiently small $\partial\upsilon_\alpha/\partial x$. A Kelvin–Voigt medium is then transformed into an elastic medium, and this occurs within the structure.

The differential equations of the one-dimensional motions of a Kelvin–Voigt medium in Lagrangian variables have the form

$$\frac{\partial\upsilon_\alpha}{\partial t} = \frac{\partial\sigma_\alpha}{\partial x} + \mu\frac{\partial^2\upsilon_\alpha}{\partial x^2}, \quad \frac{\partial u_\alpha}{\partial t} = \frac{\partial\upsilon_\alpha}{\partial x}, \quad \sigma_\alpha = \frac{\partial\Phi}{\partial u_\alpha} \tag{5.1}$$

The solution of this system in the form of a travelling wave moving together with the front with the constant velocity W is called the stationary structure of the front. We replace x by the new variable $\xi = -x + Wt$, which is positive behind the front and represents the distance from the point under consideration to the front. In the system of coordinates moving together with the front the flow is assumed to be stationary, and the equations of system (5.1) become ordinary with respect to the variable ξ. The second equation enables us to eliminate the derivatives of υ_α from the first equation, which, after a single integration, takes the form

$$\mu W\frac{du_\alpha}{d\xi} = W^2u_\alpha - \sigma_\alpha - Q_\alpha \tag{5.2}$$

Here Q_α denotes the integration constants. The equalities $u_\alpha^- = 0$ (the unstrained state) hold on the leading edge of the adhesion front. Behind the adhesion front, as $\xi \to \infty$, the solution should asymptotically approach the constant values $u_\alpha = u_\alpha^+$ and $u_\alpha = \upsilon^+$ as $du_\alpha/d\xi \to 0$. Then, for the integration constants we obtain

$$Q_\alpha = \kappa\left(\frac{W^2 - (f + (-1)^\alpha g)}{\kappa} + r^{+2}\right)u_2^+; \quad r^{+2} = u_1^{+2} + u_2^{+2}$$

Thus, the velocity of the compaction front W and the parameters u_α^+, i.e., the values of the components of the shear strains in the elastic medium behind the front, can be regarded as the given parameters of the structure problem.

6. Types of compaction fronts

The solutions of the front structure problem can be represented in the form of integral curves of a system of ordinary differential equations, which can be reduced using Eqs (5.1) and (2.1) to the form

$$\mu W\frac{du_\alpha}{d\xi} = \kappa\left(\frac{W^2 - f - (-1)^\alpha g}{\kappa} + r^2\right)u_\alpha - Q_\alpha = L_\alpha(u_1, u_2, W); \quad r^2 = u_1^2 + u_2^2 \tag{6.1}$$

The treatment will be performed in u_1, u_2, W phase space, i.e., in the u_1, u_2 phase planes for various values of W from the range $0 < W < \infty$. Note that system (6.1) is identical to the system of equations used in Refs 5 and 9 to study the structure of shock waves in the elastic medium under consideration. The structure of the shock waves was represented by the integral curve joining the singular points of system (6.1).

In the problem under consideration, among the integral curves of system (6.1) in the u_1, u_2 plane for a specified value of W, we should seek (if possible) an integral curve which leads from the state that presumably corresponds to point $O(u_\alpha = 0)$ at the time of the formation

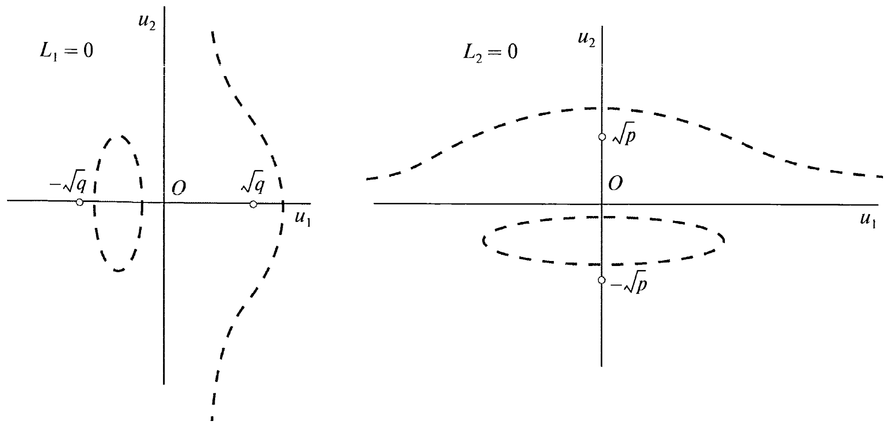

Fig. 2.

of the Kelvin–Voigt medium to a state behind the front, where $du_\alpha/d\xi = 0$ and $u_\alpha = u_\alpha^+$. Such a state can only be one of the singular points of system (6.1), in which the equalities $L_\alpha = 0$ hold simultaneously.

System of equations (5.1), (2.1) represents a hyperbolic system, to which dissipative terms in the form of second derivatives of the unknowns have been added. It has been proved for equations of this type that the type of a singular point is determined by the velocity of motion of the discontinuity relative to the characteristic velocities calculated for the state at that point.[4,5,10] In the case under consideration, from the results related to systems of equations of this type it follows that:

if $W > c_2(u_\alpha^+)$, the singular point of system (6.1) is a node with integral curves emerging from it as ξ increases,
if $c_1(u_1^+, u_2^+) < W < c_2(u_1^+, u_2^+)$, the singular point is a saddle point,
if $W < c_1(u_1^+, u_2^+)$, the singular point is a node with incoming integral curves.

The structure of the compaction front is represented in the u_1, u_2 plane by an integral curve going from the origin of coordinates to a certain singular point. If this point is a node with incoming integral curves, we will call a front that has such a structure slow (slow characteristics overtake it). If the integral curves of the front structure end at a saddle point, we will call the front fast (fast characteristics overtake it, but slow characteristics do not overtake it). We will call fronts which neither fast nor slow characteristics overtake ultrafast.

7. Isoclines of the equations of the structure and their properties

The singular points of system (6.1) in the u_1, u_2 plane lie on the intersection of the lines $L_\alpha(u_1, u_2) = 0$. These same lines are isoclines for the integral curves of system (6.1). The integral curves intersect the line $L_1 = 0$ in the direction parallel to the u_2 axis (at points of intersection where $du_1/d\xi = 0$), and they intersect the line $L_2 = 0$ in the direction parallel to the u_1 axis (where $du_2/d\xi = 0$). The signs of the functions L_1 and L_2 specify directions corresponding to an increase in ξ along the integral curves. The behaviour of the isoclines will be used to resolve the question of whether a compaction front has a structure.

The equations of the isoclines for the vertical ($\alpha = 1$) and horizontal ($\alpha = 2$) directions of the integral curves have the form

$$L_\alpha \equiv \kappa\left(\frac{W^2 - (f + (-1)^\alpha g)}{\kappa} r^2\right) u_\alpha - Q_\alpha = 0 \tag{7.1}$$

For a fixed value of W, the curves $L_1 = 0$ and $L_2 = 0$ in the u_1, u_2 phase plane have a qualitatively identical form and consist of two branches, one of which is an oval that lies on one side of one of the axes of coordinates (the u_2 axis for $L_1 = 0$ and the u_1 axis for $L_2 = 0$) and is symmetrical about the other axis. The second branch lies on the other side of this axis and goes to $\pm\infty$ as it asymptotically approaches the axis. The dashed curves shown in Fig. 2 correspond to $Q_\alpha > 0$, and we use the notations p and q, which are defined by the equalities

$$\kappa q = f - g - W^2 = c_1^{o2} - W^2, \quad \kappa p = f + g - W^2 = c_2^{o2} - W^2 \tag{7.2}$$

When the sign of Q_α changes, the corresponding isocline transforms into its mirror image about the axis serving as the asymptote. The arrangement of the branches about the axes of coordinates and the dimensions of the ovals depend both on the velocity W and the constants Q_α. For this reason, when the problem is solved, it is sometimes more convenient to use the integration constants Q_α at once instead of u_α^+.

The oval can exist as a branch of the isocline L_1 only when $q > 0$, i.e., in u_1, u_2, W phase space to the left of the point $W = c_1^o$ on the W axis. Similarly, the oval can exist as a branch of the isocline L_2 when $W < c_2^o$. Therefore, in the investigation it is useful to divide the region of variation of W into the ranges

$$0 < W < c_1^o, \quad c_1^o < W < c_2^o, \quad W > c_2^o \tag{7.3}$$

The ovals become smaller as the velocity W increases and as the integration constants Q_α increase. At large Q_α the ovals (one or both) are totally absent. For the case when $Q_\alpha > 0$ presented in Fig. 2, the disappearance of the oval of the isocline L_1 as Q_1 increases occurs when it shrinks to the point with abscissa $u_1 = -\sqrt{q/3}$ on the u_1 axis, and the disappearance of the oval of the isocline L_2 as Q_2 increases occurs when it shrinks to the point with ordinate $u_2 = -\sqrt{p/3}$ on the u_2 axis.

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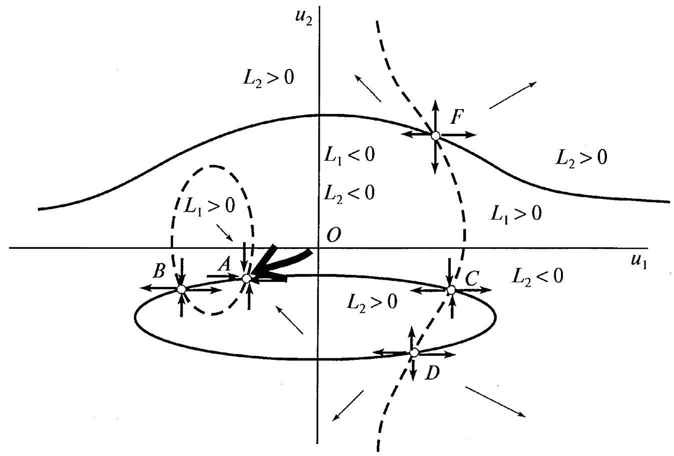

Fig. 3.

The intersection of the isoclines $L_1 = 0$ and $L_2 = 0$ in the u_1, u_2 plane gives from one to five singular points, depending on the values of the parameters indicated.

8. Sets of states behind a compaction front (an analogue of a shock adiabat) for an elastic medium with κ > 0

The goal is to determine the set of final points of the structure for all possible discontinuities with a specified initial state.

We will successively examine ranges (7.3), into which the W axis of the u_1, u_2, W phase space is divided at the values $W = c_1^o$ and $W = c_2^o$.

In the first range in (7.3) the isocline picture in the u_1, u_2 plane at small positive values of Q_α is presented in Fig. 3. The isoclines L_1 are depicted by dashed lines, and the isoclines L_2 are depicted by solid lines. In this case their intersection gives the five singular points A, B, C, D and F. The figure also indicates the signs of L_α in the different regions between the isoclines, which enable us to determine the direction of the integral curves in different parts of the u_1, u_2 plane and the types of singular points. In Fig. 3 it is indicated by arrows. In the case under consideration point A is a node with incoming lines, points B and C are saddle points, and points D and F are nodes with emerging lines.

In the region where the initial point O is located, the inequalities $L_\alpha < 0$ indicate that the integral curve emerging from point O should enter the third quadrant of the u_1, u_2 plane and reach node A. It is depicted by a thick arrow. As the integration constants Q_α are varied, the position of point A, determined by the u_1^+ and u_2^+ values, varies continuously in a certain region, whose boundaries will be determined below. The point $A(u_\alpha^+)$ remains bound by the integral curve to point O, i.e., no additional conditions in the form of equalities on the front beyond the conservation laws are required for the existence of a structure in the front, behind which the state is specified by point A (a node). In this case the integral curves cannot reach the remaining singular points from point O, i.e., this solution is unique. Note that Fig. 3 (like Fig. 2 above) corresponds to $Q_1 > 0$ and $Q_2 > 0$. In the case of other signs of the integration constants Q_α, point A (a sink) can be found in any quadrant.

When the integration constants Q_α increase (or at least one of them increases) at a fixed value of the velocity W, the ovals, as branches of the isoclines L_α, decrease in size, and their points of intersection A and B become closer and then merge into the point of contact of the isoclines. As at least one of the Q_α increases further, the points of intersection of the ovals vanish, and no integral curve going from state O to any singular point exists.

States with a point of contact between the isoclines are boundary states for the existence of a solution to the structure problem. For each fixed value of the velocity W, these points form a closed curve in the u_1, u_2 phase plane. A solution of the structure problem exists within it.

Since a pair of singular points of the structure equations can serve as states on different sides of a certain shock wave in an elastic medium, the approach of two singular points (a saddle point and a sink) followed by their merging signifies the existence of an infinitely weak, slow shock wave, whose velocity is clearly equal to the slow characteristic velocity for the state (u_1^+, u_2^+) behind the discontinuity. This means that the velocity W of a compaction front whose structure ends at the point of merging of singular points A and B is identical to the slow characteristic velocity at that point. In gas dynamics, in the theory of detonation such discontinuities, whose velocity is identical to the velocity of the small perturbations behind them, are called Jouguet waves. (This term will also be used in this paper.) Also, in the phase space the points corresponding to them, where the singular points of the structure equations merge, will be called Jouguet points.

Thus, the region where a solution of the structure problem in the u_1, u_2 plane exists at sufficiently small velocity values $W = \text{const}$ is bounded by a line consisting of the Jouguet points of the slow waves. In the three-dimensional u_1, u_2, W phase space the boundary of the region within which a solution to the structure problem exists is a surface consisting of the Jouguet points of the slow waves. We will call it the Jouguet surface. The three-dimensional set bounded by the Jouguet surface corresponds to all the possible states of the elastic medium behind the compaction front of the slow type and is, thus, part of the shock adiabat for the phase transformation front of the stream of particles into the elastic medium. For the existence of these solutions on a compaction front it is sufficient to specify only the conservation laws, and no additional conditions, such as equalities, are required. The form of the Jouguet surface will be determined below.

For all the states in the set obtained, the final point of the integral curve going from the origin of coordinates O was node A with incoming lines within the Jouguet line of the slow waves. However, it turns out that in the range of velocities W indicated there is an integral curve that goes from point O to a saddle point located outside the Jouguet line of the slow type. This is possible only when the singular point, i.e., the saddle point, lies on the u_2 axis ($u_1^+ = 0$, and, therefore, $Q_1 = 0$) and the integral curve extends from point O along the u_2 axis. In such

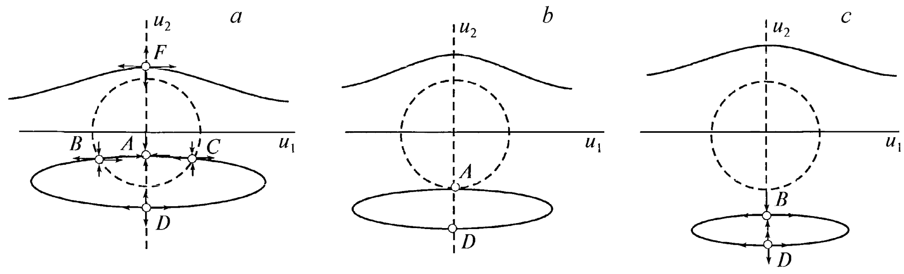

Fig. 4.

a case a branch of the isocline L_1 from the oval is transformed into a circle of radius $\sqrt{q}$ with centre at the origin of coordinates, and the other branch extends along the u_2 axis.

This situation is illustrated in Fig. 4. The isocline L_1 is represented by a dashed line, and L_2 is represented by a solid line. The branch of the latter in the form of an oval becomes smaller as Q_2 increases and then vanishes. As long as the value of Q_2 is small, the oval and the circle intersect, and sink A is a solution (Fig. 4a). At the instant when the plots of L_α are just touching, three singular points (the two saddle points and the sink) merge at $|u_2| = \sqrt{q}$, producing Jouguet point A of the slow wave (Fig. 4b), which is reachable from the initial state O and at which $W = c_1(u_\alpha^+)$. When Q_2 increases further, the oval of the isocline L_2 continues to shrink, and the integral curve from the origin of coordinates goes to saddle point B, giving a solution in the form of a fast front (Fig. 4c). The oval vanishes when $u_2 = -\sqrt{p/3}$, upon which the two singular points B (a saddle point) and D (a source) merge. The types of the singular points indicate that it is a fast Jouguet wave, the velocity of whose front is $W = c_2(u_\alpha^+)$, i.e., is equal to the fast characteristic velocity along a state behind the front. Therefore, a solution of the structure problem still exists for the segment of the u_2 axis between the two Jouguet points of the slow and fast waves

$$\sqrt{q} < |u_2| < \sqrt{p/3} \tag{8.1}$$

From definition (7.2) of p and q it is easy to see that these inequalities can hold only when

$$W > \sqrt{f - 2g} \tag{8.2}$$

i.e., a solution containing part of the u_2, W plane exists for velocities that satisfy condition (8.2).

As a result, in the first velocity range of (7.3), i.e., when $W < \sqrt{f-g}$, the three-dimensional set of states behind the compaction front obtained above for velocities that satisfy condition (8.2) should be added to the adjacent two-dimensional region lying in the u_2, W plane of the phase space between the two curves consisting of the Jouguet points of the slow and fast waves along a state behind the front.

The equation of the boundary consisting of the Jouguet points of both the slow and fast waves will be found in explicit form below. In doing so, the following must be taken into account. As was stated above, in an incompressible anisotropic elastic medium there are two types of small transverse perturbations, namely, slow and fast, which correspond to the two characteristic velocities in the system of equations of the theory of elasticity. As is seen from expressions (2.4), their values behind the front, c_1^+ and c_2^+, differ slightly due to the small degrees of non-linearity and anisotropy (adopted in this study) and can coincide as u_1^+ and u_2^+ vary. In each u_1, u_2 plane (W = const) there are two points where $c_1^+ = c_2^+$. They lie on the u_2 axis and have the ordinates

$$u_2 = \pm\sqrt{g/\kappa} \tag{8.3}$$

At such points the Jouguet surfaces can clearly intersect.

9. Jouguet lines and surfaces. The shock adiabat

The Jouguet points in a W = const plane lie where the isoclines L_α come into contact. Using Eqs (7.1), we can conclude that the set of Jouguet points in u_1, u_2, W phase space is described by the equation

$$J(u_1, u_2, W) \equiv 3(u_1^2 + u_2^2)^2 - 4\frac{f - W^2}{\kappa}(u_1^2 + u_2^2) + 2\frac{g}{\kappa}(u_2^2 - u_1^2) + pq = 0 \tag{9.1}$$

The function $J(u_1, u_2, W)$ represents a figure consisting of two surfaces, one of which contains the Jouguet points of the slow waves, while the other contains the Jouguet points of the fast waves. Figure 5 shows the intersection of the surface J = 0 with the u_2, W meridional plane, where u_1 = 0. In a cross section we obtain two curves symmetrical about the W axis

$$u_2^{(1)} = \pm\sqrt{|(W^2 - c_1^{o2})/(3\kappa)|}, \quad u_2^{(2)} = \pm\sqrt{|(W^2 - c_2^{o2})/\kappa|} \tag{9.2}$$

The first of them intersects the W axis at the point $W = c_1^o$ and consists of the Jouguet points of the slow waves, and the second curve intersects the W axis at the point $W = c_2^o$ and consists of the Jouguet points of the fast waves. Both continue in the direction of decreasing W. When $W = \sqrt{f - 2g}$, they intersect at the points $G(0, \pm\sqrt{g/\kappa})$ in Fig. 5, where the characteristic velocities c_1 and c_2 (of the slow and fast waves) switch roles. In u_1, u_2, W phase space the points G have the coordinates $0, \pm\sqrt{g/\kappa}, \sqrt{f-2g}$.

The surface J = 0 intersects each of the u_1, u_2 planes on two closed curves. It is more convenient to represent their form in a u_1, u_2 plane (W = const) in the polar system of coordinates

$$u_1 = r\cos\varphi, \quad u_2 = r\sin\varphi, \quad r^2 = u_1^2 + u_2^2$$

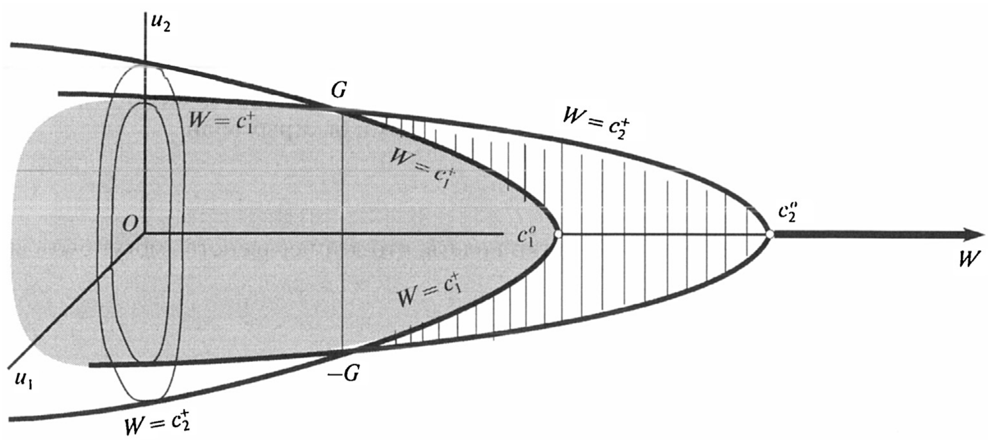

Fig. 5.

In this system Eq. (9.1) is a biquadratic equation in r. Its solutions have the form

$$r_{1,2}^2 = \frac{1}{3\kappa}\left[2(f-W^2) + g\cos 2\varphi \pm \sqrt{(f-W^2+2g\cos 2\varphi)^2 + 3g^2\sin^2 2\varphi}\right] \quad (9.3)$$

and are represented in each u_1, u_2 plane (W = const) by two closed curves that encircle the origin of coordinates. Both curves are symmetrical about the u_1 and u_2 axes, and one of them always lies within the other. The equalities $W = c_1^+$ and $W = c_2^+$ hold on the inner and outer curves, respectively. The curves have two common points only in the case when the radicand in formula (9.3) vanishes, i.e., when $\varphi = \pm/2$ and $W = f - 2g$. This value of W satisfies the condition under consideration $W < c_1^o$. The points of intersection lie on the u_2 axis, have ordinates (8.3) in the plane $W = \sqrt{f-2g}$ and are the points described above where the characteristic velocities are equal: $c_1^+ = c_2^+$. In Fig. 5 they are denoted by G and $-G$.

Thus, in the u_1, u_2, W phase space the set of states behind a compaction front (an analogue of a shock adiabat) at $W < c_1^o$ contains a three-dimensional region within the surface consisting of the Jouguet points of the slow waves. In Fig. 5 this region is shaded.

In addition, for velocities that satisfy condition (8.2) this set also includes a two-dimensional region in the form of part of the u_2, W plane between the two branches of (9.2) for the slow and fast Jouguet points. As was shown above, such a solution exists only for values of u_2 in range (8.1), which corresponds to condition (8.2), i.e., in the u, W plane to the right of the points where $c_1 = c_2$. In Fig. 5 this part of the shock adiabat is covered with vertical hatching. In a large-scale treatment for the evolutionarity of discontinuities of this type, the additional condition $u_1 = 0$ must be imposed on the front along with the conservation laws. In addition, double inequality (7.3), which restricts the region of final states behind the structures of these fronts in the plane indicated, should be taken into account.

The isoclines and integral curves in the second range of (7.3), where $q < 0$ and the branch in the form of the oval of the isocline L_1 (the dashed curve in Fig. 3) is absent, is investigated similarly, and for arbitrary values of Q_α no solutions in the form of a structure exist except in the case of $u_1 = 0$, $Q_1 = 0$, in which a picture similar to the one depicted in Fig. 4, is obtained. The structure exists in a two-dimensional region in the u_2, W meridional plane between the lines

$$u_2^{(2)} = \pm\sqrt{p/(3\kappa)}$$

which consist of the Jougnet points of the fast waves. This region is a continuation of the two-dimensional part of the region of states behind the front described for $W < c_1^o$ (Fig. 5) and is also marked by vertical hatching.

When $W > c_2^o$, both of the isoclines L_α in the u_1, u_2 plane consist of only one branch, and there is no branch in the form of ovals. The only intersection point is a source, which integral curves cannot enter. In this case there is no solution in the form of a structure for any u_α values. Within the compaction front no elastic strains appear, and behind the front they should remain equal to zero: $u_\alpha = 0$. As a result, the region of states behind the front in u_1, u_2, W phase space at the velocity values $W > c_2^o$ consists of a one-dimensional set in the form of the segment of the W axis demarcated in Fig. 5 by the thick line. This means that when the compaction front is treated as a discontinuity surface under the large-scale approach, the equalities $u_\alpha = 0$ should be regarded as two additional conditions (to the conservation laws) on the discontinuity, which ensure the evolutionarity of ultrafast adhesion fronts, at which $W > c_2^o$.

In conclusion we mention the problem of a compaction front in the case when the anisotropy parameter g in expression (2.1) is not small. Then, for small u_α the quadratic terms containing the constants f and g are of the same order of magnitude, and the term containing κ, which corresponds to the non-linearity, can be neglected. In this case the characteristic velocities c_α take the constant values $c_\alpha = c_\alpha^o$, which divide the region of variation of W into the three ranges (7.3). The variation of u_α in this case can be obtained from the versions considered by taking the limit as $\kappa \to 0$. It is easy to see that $u_\alpha^+ = 0$ when $W > c_2^o$, that $u_1^+ = 0$ and u_2^+ is arbitrary when $W < c_2^o$ and that both u_α^+ are arbitrary when $W < c_1^o$.

Finally, we mention that the values of the transverse velocity components υ_α^+ are found from the values of u_α^+ using relations (3.1). Since the primary term in the expression for the elastic potential is the first term, the character of the variation of the velocities in the adhesion front qualitatively repeats the variation of the u_α values.

10. The self-similar wave problem in a half-space

The self-similar wave problem in a half-space, which is often called the piston problem, refers here to the problem in which the medium formed behind a compaction front is adjacent to a plane, i.e., a piston, which moves with a specified constant velocity relative to the homogeneous medium ahead of the front. It is assumed here that all three components of the piston velocity are identical to the corresponding velocity components of the elastic medium adjacent to it. The solution of the piston problem contains a compaction front and may contain elastic waves that move through the elastic medium behind this front with small velocities.

In the "hyperbolic" approximation considered below, in which the dissipative effects are neglected in the continuous processes and the discontinuities are assumed to be infinitely thin (however, the relations on them are guaranteed by the fact that they have a structure), the problem is self-similar. All the quantities depend on x/t, and the solution consists of one or several waves separated by regions with constant parameters. Since all waves move relative to the medium, an expanding region with constant parameters that satisfy the conditions on the piston is always adjacent to the piston.

On the piston, beside the three velocity components, three stress components can be specified. In the case of an incompressible medium, the normal component of the stresses is constant everywhere behind the compaction front and is determined immediately behind it according to the conservation of the normal component of the momentum, and it will not be subject to further study. The normal stress on the piston is determined uniquely by the velocity of the front relative to the specified flow of particles and the value W of the relative velocity of the elastic medium formed relative to the front.

According to equalities (2.2), the shear stresses are expressed in terms of u_α; therefore, below we will assume that the values of $u_\alpha = u_\alpha^*$ are given and that they are arbitrary in a certain range.

First consider range (8.2). As was already noted in Section 7, the origin of coordinates in the u_1, u_2 plane belongs to the region of u_α values behind slow compaction fronts, which is bounded by the Jouguet curve for the slow fronts (only slow fronts are possible under condition (8.2)). In the solution of the problem behind a slow front which is not a Jouguet front, the homogeneous state adjacent to the piston follows.

If the state u_α behind the front belongs to the boundary of the region of states behind slow fronts (9.1) in the u_1, u_2 plane, the Jouguet equality $W = c_1(u_1, u_2)$ holds on this front according to the foregoing. Behind such a front, in the space adjacent to it, a slow non-overturning Riemann wave can propagate in the elastic medium.[5,6] If the integral curves of the slow Riemann waves are drawn in the u_1, u_2 plane from each point on the boundary (the Jouguet line), they fill the entire u_1, u_2 plane without intersecting. Thus, if the point $(u_1^*,\ u_2^*)$ lies outside the region of states behind a slow compaction front, a slow Riemann wave (which corresponds to the characteristic velocity $c_1(u_1, u_2)$) is closest to the piston. In this wave the u_α values vary from the u_α^* values in the region adjacent to the piston, to the values at a certain point on the Jouguet line which represents the states immediately behind the slow Jouguet compaction front.

If the value of W satisfies the inequality

$$\sqrt{f-2g} < W < c_1^o$$

the states behind the compaction front in the u_1, u_2 plane, as in the preceding case, contain a region of values of the states behind the slow fronts, which is restricted by the Jouguet line. Nevertheless, as was shown, on the u_2 axis there is a segment, whose points represent states behind the fast fronts. The ends of this segment $u_2^{(\alpha)}$ satisfy the conditions $c_2(0,\ u_2^{(\alpha)}) = W$. On the u_2 axis there is also a point with ordinate $u_2^{(3)} = \sqrt{g/\kappa}$, in which $c_1(0, u_2) = c_2(0, u_2)$, and its position does not depend on the value of the velocity W. In the case considered above, in which condition (8.2) is satisfied, this point was always found within the region of the states behind the slow fronts and consequently did not play any role. In the range of W values under consideration this point lies outside the region of the slow compaction waves and participates[5] in the construction of solutions with elastic waves. The $u_2^{(1)}$, $u_2^{(2)}$, $u_2^{(3)}$ values satisfy the inequalities

$$u_2^{(1)} < u_2^{(2)} < u_2^{(3)}$$

The solution of a problem similar to the preceding one can consist of only one compaction front, if the point $(u_1^*,\ u_2^*)$ belongs to the set of states behind slow or fast fronts described above. If $u_1^* = 0$, and the ordinate u_2^* lies between $u_2^{(2)}$ and $u_2^{(3)}$, the solution consists of a fast Jouguet front with a state behind the front $u_1 = 0$, $u_2 = u_2^{(2)}$. Behind this front there is an expanding fast elastic Riemann wave,[5] behind which the state is represented by any point on the u_2 axis from the range $u_2^{(2)} < u_2 < u_2^{(3)}$ on the u_2 axis (the range $-u_2^{(2)} < u_2 < u_2^{(3)}$ on the u_2 axis represents the integral curve of the fast elastic Riemann wave). Beside the waves enumerated, a slow elastic Riemann wave, which has a smaller velocity and is closer to the piston, may be present in the solution of the problem. In it the u_α values vary from the u_α^* values on its trailing edge to the values on a slow Jouguet line or to the values behind the fast front on the segment

$$u_2^{(2)} \le u_2 \le u_2^{(3)}$$

of the u_2 axis. The integral curves of the slow elastic Riemann waves[5] are depicted in Fig. 6.

Construction of the solution is most complicated when the u_1^* value is fairly small and $u_2^* > u_2^{(3)}$. In this case upon motion along a slow Riemann wave (which is closest to the piston) away from the piston, the u_α values vary along the integral curve from the u_α^* values until the point (u_1, u_2) is no longer on the u_2 axis between the $u_2^{(2)}$ and $u_2^{(3)}$ values. Then (after a region with constant $u_1 = 0$ and u_2) a fast elastic Riemann wave follows, in which u_2 decreases from the trailing edge of the wave to the leading edge and takes the value $u_2^{(2)}$ on its leading edge. The leading edge of this wave is adjacent to the fast Jouguet compaction front, in which

$$u_1^+ = 0, \quad u_2^+ = u_2^{(2)}, \quad u_1^- = 0, \quad u_2^- = 0$$

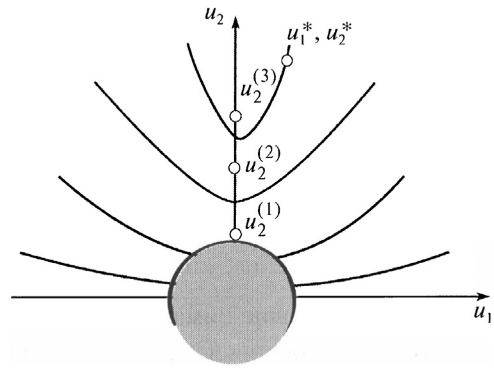

Fig. 6.

If the velocity W lies in the second range in (7.3), then unlike the preceding case, the region corresponding to the slow fronts vanishes, while on the u_2 axis the point with the ordinate $u_2^{(1)}$ vanishes, and the points with the ordinates $u_2^{(2)}$ and $u_2^{(3)}$ remain. The set of solutions is reorganized, and the solutions containing the slow compaction fronts disappear from it.

In the third range in (7.3) the points on the u_2 axis with ordinates $u_2^{(\alpha)}$ are absent, and only the point with the ordinate $u_2^{(3)}$ remains. In the ultrafast front the u_α values do not vary (they remain equal to zero), but there is variation of u_α in the fast and slow elastic waves, which are separated from one another and from the compaction front by regions with constant parameters.

Thus, it has been shown that a solution of the piston problem exists and is uniquely defined for all compaction front velocities W and all possible shear strains values u_α^* specified on the piston. Finally, the u_α^* values are assumed to be small enough for the representation of the elastic potential by expansion (2.1) to be valid.

11. Conclusion

For non-linear elastic media with a stiffness that decreases as the strains increase ($\kappa > 0$) under the condition of small strains, the structure of the fronts for the formation of an elastic medium from a stream of free particles was investigated under the condition that the structure of the front is described by the Kelvin–Voigt equations. The shock adiabat, i.e., the set of states behind all possible fronts moving towards a specified stream of particles, was constructed under the assumption that the fronts have a structure. In the u_1, u_2, W space of values, the shock adiabat, qualitatively depicted in Fig. 5, consists of parts of different dimensionality. This determines the presence or absence of additional relations for the variation of a quantity on the fronts. The requirement for the existence of a structure in the case of slow fronts does not place restrictions on the variation of quantities in the form of conditions expressed as equalities. Apart from the relations in the form of the equalities $u_1 = 0$ (fast fronts) or $u_1 = 0$, $u_2 = 0$ (ultrafast fronts), there are relations in the form of inequalities that restrict the variation of quantities in the fronts of different types. Figure 5 qualitatively describes these restrictions.

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