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Genesis of d'Alembert's paradox and analytical elaboration of the drag problem

G. Grimberg[a], W. Pauls[b,c], U. Frisch[b,*]

[a] Instituto de Matemática, Universidade Federal do Rio de Janeiro (IM-UFRJ), Brazil
[b] Labor. Cassiopée, UNSA, CNRS, OCA, BP 4229, 06304 Nice Cedex 4, France
[c] Fakultät für Physik, Universität Bielefeld, Universitätsstraße 25, 33615 Bielefeld, Germany

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Abstract

We show that the issue of the drag exerted by an incompressible fluid on a body in uniform motion has played a major role in the early development of fluid dynamics. In 1745 Euler came close, technically, to proving the vanishing of the drag for a body of arbitrary shape; for this he exploited and significantly extended the existing ideas on decomposing the flow into thin fillets; he did not however have a correct picture of the global structure of the flow around a body. Borda in 1766 showed that the principle of live forces implied the vanishing of the drag and should thus be inapplicable to the problem. After having at first refused the possibility of a vanishing drag, d'Alembert in 1768 established the paradox, but only for bodies with a head–tail symmetry. A full understanding of the paradox, as due to the neglect of viscous forces, had to wait until the work of Saint-Venant in 1846.

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1. Introduction

The first hint of d'Alembert's paradox – the vanishing of the drag for a solid body surrounded by a steadily moving ideal incompressible fluid – had appeared even before the analytical description of the flow of a "perfect liquid"[1] was solidly established. Leonhard Euler in 1745, Jean le Rond d'Alembert in 1749 and Jean-Charles Borda in 1766 came actually very close to formulating the paradox, using momentum balance (in an implicit way) or energy conservation arguments, which actually predate its modern proofs.[2] D'Alembert in 1768 was the first to recognize the paradox as such, although in a somewhat special case. Similarly to Euler and Borda, his reasoning did not employ the equations of motion directly, but nevertheless used a fully constituted formulation of the laws of hydrodynamics, and exploited the symmetries he had assumed for the problem. A general formulation of d'Alembert's paradox for bodies of an arbitrary shape was given in 1846 by Adhémar Barré de Saint-Venant, who pointed out that the vanishing of the drag can be due to not taking into account viscosity. Other explanations of the paradox involve unsteady solutions, presenting for example a wake, as discussed by Birkhoff.[3]

Since the early derivations of the paradox did not rely on Euler's equation of ideal fluid flow, it was not immediately recognized that the idealized notion of an inviscid fluid motion was here conflicting with the physical reality. The difficulties encountered in the theoretical treatment of the drag problem were attributed to the lack of appropriate analytical tools rather than to any hypothetical flaws in the theory. In spite of the great achievements of Daniel and Johann Bernoulli, of d'Alembert

* Corresponding author. Tel.: +33 4 92003035; fax: +33 4 92003058.
E-mail addresses: gerard.emile@terra.com.br (G. Grimberg), uriel@obs-nice.fr (U. Frisch).

[1] Kelvin's name of an incompressible inviscid fluid.

[2] See, e.g. Serrin, 1959 and Landau and Lifshitz, 1987.

[3] Euler, 1745; D' Alembert, [1749]; Borda, 1766; Saint-Venant, 1846, 1847; Birkhoff, 1950: Chap. 1, §9.

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and of Euler[4] the theory of hydrodynamics seemed beset with insurmountable technical difficulties; to the contemporaries it thus appeared of little help, as far as practical applications were concerned. There was a dichotomy between, on the one hand, experiments and the everyday experience and, on the other hand the eighteenth century's limited understanding of the nature of fluids and of the theory of fluid motion. This dichotomy is one of the reasons why neither Euler nor Borda nor the early d'Alembert were able to recognize and to accept the possibility of a paradox.

We shall also see, how the problem setting became more and more elaborated in the course of time. Euler, in his early work on the drag problem appeals to several physical examples of quite different nature, such as that of ships navigating at sea and of bullets flying through the air. D'Alembert's 1768 formulation of the drag paradox is concrete, precise and much more mathematical (in the modern sense of the word) than Euler's early work. This is how d'Alembert was able to show – with much disregard for what experiments or (sometimes irrelevant) physical intuition might suggest – that the framework of inviscid fluid motion necessarily leads to a paradox.

For the convenience of the reader we begin, in Section 2, by recalling the modern proofs of d'Alembert's paradox: one proof – somewhat reminiscent of the arguments in Euler's 1745 work – relies on the calculation of the momentum balance, the other one – connected with Borda's 1766 paper – uses conservation of energy. In Section 3 we describe Euler's first attempt, in 1745, to calculate the drag acting on a body in a steady flow using a modification of a method previously introduced by D. Bernoulli.[5] In Section 4 we discuss d'Alembert's 1749 analysis of the resistance of fluids. In Section 5 we review Euler's contributions to the drag problem made after he had established the equations of motions for ideal fluid flow. Section 6 is devoted to Borda's arguments against the use of a live-force (energy conservation) argument for this problem. In Sections 7 and 8 we discuss d'Alembert's and Saint-Venant's formulation of the paradox. In Section 9 we give the conclusions.

Finally, we mention here something which would hardly be necessary if we were publishing in a journal specialized in the history of science: the material we are covering has already been discussed several times, in particular by such towering figures as Saint-Venant and Truesdell.[6] Our contributions can only be considered incremental, even if, occasionally, we disagree with our predecessors.

2. Modern approaches to d'Alembert's paradox

Let us consider a solid body K in a steady potential flow with uniform velocity U at infinity. In the standard derivation of the vanishing of the drag[7] one proceeds as follows: Let Ω be the domain bounded in the interior by the body K and in the exterior by a sphere S with radius R (eventually, $R \to \infty$). The force acting upon K is calculated by writing a momentum balance, starting from the steady incompressible 3D Euler equation

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p, \qquad \nabla \cdot \boldsymbol{v} = 0. \tag{1}$$

The contribution of the pressure term gives the sum of the force acting on the body K and of the force exerted by the pressure on the sphere S. It may be shown, using the potential character of the velocity field, that the latter force vanishes in the limit $R \to \infty$. The contribution of the advection term can be written as the flux of momentum through the surface of the domain Ω: the flux through the boundary of K vanishes because of the boundary condition $\boldsymbol{v} \cdot \boldsymbol{n} = 0$; the flux through the surface of S vanishes because the velocity field is asymptotically uniform ($v \simeq U$ for $R \to \infty$). From all this it follows that the force on the body vanishes. This approach proves the vanishing of both the drag and the lift.[8]

Alternatively, one can use energy conservation to show the vanishing of the drag.[9] Roughly, the argument is that the work of the drag force, due to the motion with velocity U, should be balanced by either a dissipation of kinetic energy (impossible in ideal flow when it is sufficiently smooth) or by a flow to infinity of kinetic energy, which is also ruled out for potential flow. This argument shows only the vanishing of the drag.

A more detailed presentation of such arguments may be found in the book by Darrigol.[10]

In the following we shall see that many technical aspects of these two modern approaches were actually discovered around the middle of the eighteenth century.

3. Euler and the new principles of Gunnery (1745)

In 1745 Euler published a German translation of Robins' book "New Principles of Gunnery" supplemented by a series of remarks whose total amount actually makes up the double of the original volume. In the third remark of the first proposition (Dritte Anmerkung zum ersten Satz) of the 2nd Chapter Euler attempts to calculate the drag on a body at rest surrounded by a steadily moving incompressible fluid.[11]

In 1745 the general equations governing ideal incompressible fluid flow were still unknown. Nevertheless, Euler managed the remarkable feat of correctly calculating the force acting on an element of a 2D steady flow around a solid body. For this, as we shall see, he borrowed and extended the results obtained by D. Bernoulli a few years earlier.[12]

4 See, e.g., Darrigol, 2005; Darrigol and Frisch, 2008.
5 Bernoulli, 1736.
6 Truesdell, 1954; Saint- Venant [1888].
7 See, e.g., Serrin, 1959.
8 The lift need not vanish if there is circulation.
9 See, e.g., Landau and Lifshitz, 1987: §11.
10 Darrigol, 2005: Appendix A.
11 For the German original of the third remark, cf. Euler, 1745: 259–270 (of *Opera omnia* which we shall use for giving page references); an English version, taken from Hugh Brown's 1777 translation is available at www.oca.eu/etc7/EE250/texts/euler1745.pdf. We shall sometimes use our own translations.
12 Bernoulli, 1736, 1738.

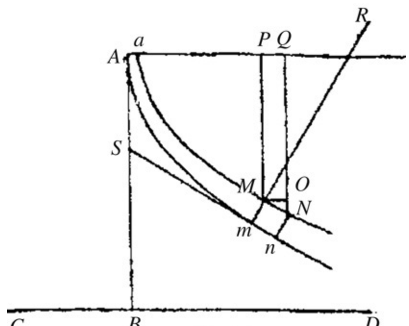

Fig. 1. Figure 14 of Euler, 1745: 263: this figure represents a fillet of fluid aAMm, deflected by the solid body, but the shape of the body is not fully specified.

Euler begins by noting that instead of calculating the drag acting on a body moving in a fluid one can calculate the drag acting on a resting body immersed in a moving fluid. Thus, he considers a fluid moving into the direction AB[13] (cf. Figs. 1 and 3), past a solid body CD.[14] Then Euler continues by describing the motion of fluid particles and establishes a relation between the trajectory and velocity of each fluid particle and the force which is acting on this particle. He observes that, instead of determining the force on the body, one can evaluate the reaction on the fluid:

But since all parts of the fluid, as they approach the body, are deflected and change both their speed and direction [of motion], the body has to experience a force of strength equal to that needed for this change in speed as well as direction of the particles.[15]

Thus, one has to determine the force which is applied at each point of the fluid. Euler chooses a fillet[16] AaMm of fluid with an infinitesimal width and observes that the velocity[17] v of the particles passing through the section Mm is inversely proportional to its (infinitesimal) width $Mm = \delta z$; so that $v\,\delta z = v_0\,\delta z_0$, where $\delta z_0 =$ Aa and v_0 are the width of the fillet and the velocity at the reference point A.[18] For later reference let us call this relation mass conservation. Euler assumes that the particles passing through the section Aa follow the fillet AaMm. This is equivalent to assuming that the velocity in each section Mm along the trajectory depends only on the location of the point M and not on time, in modern terms a stationary flow. Here the concepts of streamline and of stationarity in two dimensions appear for the first time explicitly.

With the above assumption, Euler defines

$$AP = x, \qquad PQ = \mathrm{d}x, \qquad PM = y, \qquad ON = \mathrm{d}y,$$
$$p = \mathrm{d}y/\mathrm{d}x, \qquad MN = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}x\sqrt{1+p^2}. \tag{2}$$

Since the force exerted by the body on the fluid is oriented upward, we prefer orienting the vertical axis upward. Hence y and p will be negative in what follows. Otherwise we shall mostly follow Euler's notation. Euler calculates the normal and tangential components, $\mathrm{d}F_\mathrm{N}$ and $\mathrm{d}F_\mathrm{T}$, of the infinitesimal force acting on the element of fluid fillet MNnm (see Fig. 1).[19]

With the assumed unit density, the mass of fluid in MNnm is

$$\delta z \times MN = \delta z \mathrm{d}x\sqrt{1+p^2}. \tag{3}$$

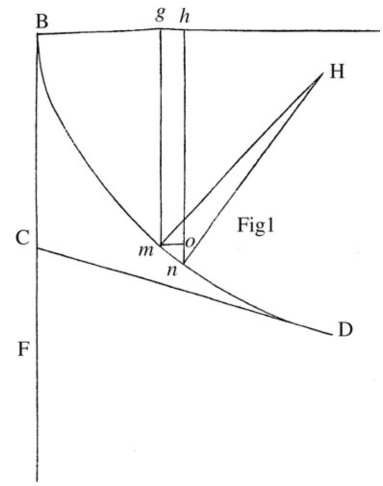

Fig. 2. Figure 1 of Bernoulli, 1736. A centripetal argument is used to calculate the normal force acting along a fillet of fluid represented here just by the curve BD (changes in width are ignored).

The normal acceleration $\mathrm{d}F_\mathrm{N}$ in the direction MR is calculated by Euler as a centripetal acceleration, i.e., given by the product of the square of the velocity v^2 and the curvature $(1+p^2)^{\frac{3}{2}}\mathrm{d}p/\mathrm{d}x$. Euler may here be following D. Bernoulli.[20] The latter, in a paper concerned among other things with jets impacting on a plane, had developed an analogy between an element of fluid following a curved streamline and a point mass on a curved trajectory (cf. Fig. 2). Multiplying the acceleration by the elementary mass and using mass conservation,[21] Euler then obtains

$$\mathrm{d}F_\mathrm{N} = v_0\delta z_0 v\mathrm{d}p/(1+p^2), \tag{4}$$

in which the velocity v along the fillet is considered to be a function of the slope p.

[13] Here, contrary to the usage in Eulers' memoir, all geometrical points will be denoted by roman letters, leaving italics for algebraic quantities.

[14] These are Euler's own words; examination of various of his figures and of the scientific context shows that the body extends below CD and, perhaps also above.

[15] Euler, 1745: 263. Weil aber alle Theile der flüßigen Materie, so bald sich dieselben dem Körper nahen, genöthiget werden auszuweichen, und so wohl ihre Geschwindigkeit, als ihre Richtung zu verändern, so muß der Körper eine eben so große Kraft empfinden, als zu dieser Veränderung so wohl in der Geschwindigkeit, als der Richtung der Theilchen, erfordert wird.

[16] Euler uses the word "Canal" (channel).

[17] Following early eighteenth century notation, Euler represents a velocity by the corresponding height of free fall to achieve the given velocity, starting a rest; in modern notation this would be $\sqrt{2gh}$. In the 1745 paper Euler takes mostly $g = 1$ – but occasionally $g = 1/2$ – and denotes the height by v. In order not to confuse the reader, we shall here partially modernize the notation and in particular denote the velocity by v.

[18] Euler denotes our δz, δz_0 and v_0 by z, a and $\sqrt{2b}$, respectively.

[19] The notation $\mathrm{d}F_\mathrm{N}$ and $\mathrm{d}F_T$ is ours.

[20] Bernoulli, 1736 and 1738: Section XIII, §13.

[21] Bernoulli, 1738: 287 assumed a fillet of uniform width (*fistulam implantatam esse uniformis quidem amplitudinis*) and thus did not use mass conservation to relate the varying width and velocity.

To obtain the tangential force $\mathrm{d}F_{\mathrm{T}}$ in the direction mS on the element of fillet, Euler writes

$$\delta z \mathrm{d}x\sqrt{1+p^2}\mathrm{d}(v^2/2) = -\mathrm{d}F_{\mathrm{T}}\,\mathrm{d}x\sqrt{1+p^2}, \tag{5}$$

and thus

$$\mathrm{d}F_{\mathrm{T}} = -\delta z \mathrm{d}(v^2/2) = -\delta z_0(v_0/v)\mathrm{d}(v^2/2). \tag{6}$$

For the case of Fig. 1 the force is oriented in the direction mS, because the fluid is moving more slowly at N than at M. Euler does not elaborate on how he derives (5) but this seems typically a “live-force” argument of a kind frequently used at that time, for example by the Bernoullis.[22] Indeed the l.h.s. is the variation of the live force (kinetic energy) and the r.h.s. is what we would now call the work of the tangential force per unit mass.

So as to later determine the drag, that is the force on the body in the vertical direction, Euler adds these normal and tangential elementary forces, projected onto the vertical axis oriented in the direction BA. He thus obtains the following elementary vertical force on the fluid:

$$\mathrm{d}F_{\mathrm{BA}} = v_0\delta z_0\left(\frac{v\mathrm{d}p}{(1+p^2)^{\frac{3}{2}}} + \frac{p\mathrm{d}v}{\sqrt{1+p^2}}\right). \tag{7}$$

Here a “miracle” happens: the r.h.s. of (7) is the exact differential of

$$v_0\delta z_0\left(\frac{vp}{\sqrt{1+p^2}}\right). \tag{8}$$

Finding the exact form of the function $v(p)$, as we now know, requires the solution of a non-trivial boundary value problem. The exact form does however not matter for the integrability property and – from a modern perspective – can be related to the global momentum conservation property of the Euler equation. In 1745 Euler did not comment on the miracle. It is worth stressing that it does not survive if any error is made regarding the numerical factors appearing in the normal and tangential forces.

Euler is now able to exactly integrate the elementary force along a fillet from its starting point A, assumed to be far upstream ($p=-\infty$), to a point m with a finite slope p. Noting that $-p/\sqrt{1+p^2}$ is the cosine of the angle MSB, he obtains the following force on the body, due to the fillet:

$$F_{\mathrm{AB}} = -v_0^2\delta z_0\left(1-\frac{v}{v_0}\cos \mathrm{MSB}\right). \tag{9}$$

Note that this is a force from a given fillet of infinitesimal width which must still be integrated over a set of fillets encompassing the whole fluid. More important here is where to terminate the fillet. It is clear that the relevant fillets start far upstream in the vertical direction; but where do they lead after having come close to the solid body? Euler considers various possibilities, such as a 90° deflection. He then envisages a very interesting case:

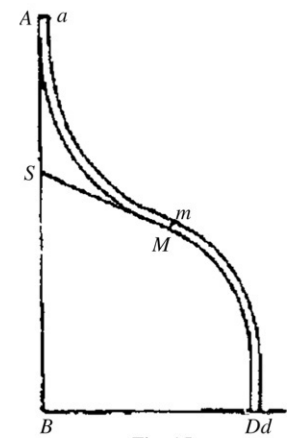

Fig. 15

Fig. 3. Figure 15 of Euler, 1745: 268 from which he tries to explain that the drag should be calculated using only the portion AM of the fillet.

It remains therefore only to fix upon the point which is to be esteemed the last of the canal. If we go so far that the fluid may pass by the body, and attain its first direction and velocity then shall $\delta z = \delta z_0$, and the angle mSB vanish, and therefore its cosine = 1, then shall the force acting on the body in the direction AB $= -v_0^2\delta z_0(1-1) = 0$, and the body suffers no resistance. [23]

From a technical point of view Euler’s 1745 derivation of the vanishing of the drag force has many features of the modern proof. However Euler refuses here to see a paradoxical property of the model of ideal fluid flow (for which the equations are not even completely formulated). He accepts the possibility that the vanishing of the drag applies to certain exotic fluids which are “infinitely fluid ... and also compressed by an infinite force”[24] such as the hypothetical ether surrounding celestial bodies (called by him “subtle heavenly material”), but he firmly rejects it for water and air. Indeed, immediately after the previous citation he writes:

Hence it appears, that for air or water, we are not to take the point of the canal for last, where the motion behind the body corresponds exactly with that at the beginning of the canal. [25]

Euler then explains why in his opinion the “last point” should not at all be taken far downstream, but rather near the inflection point M where the angle MSB achieves its maximum value, as shown in Fig. 3.[26] As pointed out to us by Olivier Darrigol, in Euler’s opinion the portion AM of the canal AD is the only one that exerts a force on the body, the alleged reason

[22] Cf., e.g., Darrigol, 2005: Chap. 1.

[23] Euler, 1745: 267. Hier kömmt es also nur darauf an, wo das Ende des Canals angenommen werden soll. Geht man so weit, biß die flüßige Materie um den Körper völlig vorbey geflossen, und ihren vorigen Lauf wiederum erlanget hat, so wird ..., und der Winkel mSB verschwindet, dahero der Cosinus desselben = 1 wird. In diesem Fall würde also die auf den Körper nach der Direction AB würkende Kraft ... und der Körper litte gar keinen Wiederstand.

[24] Euler, 1745: 268–269. ... unendlich flüßig ... und von einer unendlichen Kraft zusammen gedruckt ...

[25] Euler, 1745: 267. Woraus erhellet, daß man für Wasser und Luft nicht denjenigen Punkt des Canals, wo die Bewegung hinter dem Körper mit der ersten wiederum völlig übereinkommt, für den letzten annehmen könne.

[26] Truesdell, 1954: XL writes that “Euler supposes that the oncoming fluid turns away from the axis, leaving a dead-water region ahead of the body”; actually, Euler does not assume any dead-water region in his Third Remark.

being that the force caused by the deflection in the portion MD is not directed toward the body:

> The other part DM produces a force which is opposite to the first, and would cause the body to move back in the direction BA. Now, as only a true pressure [a positive one] can set a body into motion, the latter force can only act on the body insofar as the pressure of the fluid matter from behind is strong enough to move the body forwards .[27]

Hence he departed from strict dynamical reasoning to follow a dubious intuition of the transfer of force through the fluid.[28]

To sum up, Euler performed a real *tour de force* by deriving the correct expression for the force on a fillet of fluid without having the equations of motion but practically he was not able to reach much beyond Newton's impact theory when considering the global interaction between the fluid and the body.

4. d'Alembert and the treatise on the resistance of fluids (1749)

In a treatise[29] written for the prize of the Berlin Academy of 1749 whose subject was the determination of the drag a flow exerts upon a body, d'Alembert gives a description of the motion of the fluid analogous to that of Euler. It is not clear if d'Alembert knew about Euler's "Commentary on Gunnery". As noted by Truesdell,[30] some figures in d'Alembert's treatise are rather similar to those found in the Gunnery but there are also arguments in the Gunnery which would have allowed d'Alembert, had he been aware of them, to extend his 1768 paradox to cases not possessing the head–tail symmetry he had to assume. Anyway, d'Alembert was fully aware of D. Bernoulli's work on jet impact in which, as we already pointed out, a similar figure is found.

In the treatise d'Alembert described the motion of an incompressible fluid in uniform motion at large distance, interacting with a localized axisymmetric body. He observed that the streamlines and the velocity of the fluid at each point in space are time-independent. The velocity a of the fluid far upstream of the body is directed along the axis of symmetry (which he takes for the abscissa); the other axis is chosen to be perpendicular to this direction. In this frame a point M of the fluid is characterized by the cylindrical coordinates (x, z) and the corresponding velocity has the components av_x and av_z.[31]

D'Alembert's first aim is to derive the partial differential equations which determine the motion of the fluid, and the appropriate boundary conditions with which they must be supplemented. He observed that, in order to determine the drag on the body, one must first determine

> ...the pressure of the fillet of Fluid which glides immediately on the surface of the body. For this it is necessary to know the velocity of the particles of the fillet. [32]

By considering the motion of fluid particles during an infinitesimal time interval, d'Alembert is able to find the expressions of the two components of the force acting on an element of fluid:

$$\gamma_z = a^2\left(-v_x\frac{\partial v_z}{\partial x} - v_z\frac{\partial v_z}{\partial z}\right), \tag{10}$$

and

$$\gamma_x = a^2\left(-v_x\frac{\partial v_x}{\partial x} - v_z\frac{\partial v_x}{\partial z}\right). \tag{11}$$

From this d'Alembert derived for the first time the partial differential equations for axisymmetric, steady, incompressible and irrotational flow, but he does not use such equations in considering the problem of "fluid resistance".[33]

How does d'Alembert calculate the drag? From an assumption about the continuity of the velocity he infers, contrary to Euler, that there must be a zone of stagnating fluid in front of the body and behind it, bordered by the streamline TFMDLa which attaches to the body at M and detaches at L (see Fig. 4).[34]

In his calculation of the drag d'Alembert used an approach which differed from that of Euler in the Gunnery: instead of calculating the balance of forces acting on the fluid he considered the pressure force exerted on the body by the fluid fillet in immediate contact with it. D'Alembert noted first that, for each surface element of the body, the force exerted by the fluid particles is perpendicular to this surface, because of the vanishing of the tangential forces, characterizing the flow of an ideal fluid.[35]

In conformity with Bernoulli's law, d'Alembert expressed the pressure along the body as $a^2(1-v_x^2-v_z^2)$. With ds denoting the element of curvilinear length along the sections of the body by an axial plane such as that of Fig. 4, the infinitesimal element of surface of revolution of the body upon which this pressure is acting is $2\pi z \mathrm{d}s$. The component along the axis of the pressure force exerted is

$$2\pi a^2(1 - v_x^2 - v_z^2)z\mathrm{d}z. \tag{12}$$

Further integration along the profile AMDLC yields the vertical component of the drag.

Then came a very important remark. D'Alembert noted that in the case of a body which is not only axisymmetric but has

[27] Euler, 1745: 268. Aus dem andern Theil DM aber ensteht eine Kraft, welche jener entgegen ist, und von welcher des Körper nach der Direction BA zurück gezogen werden sollte. Da nun kein Körper anders, als durch einen würklichen Druck in Bewegung gesetzt werden kann, so kann auch die letztere Kraft nur in so ferne auf den Körper würken, als der Druck der flüßigen Materie von hinten stark genug ist, den Körper vorwärts zu stossen.

[28] Darrigol, private communication, 2007.

[29] D'Alembert, [1749], 1752.

[30] Truesdell, 1954: LII.

[31] D'Alembert uses a similarity argument to prove that the velocity field around a body of a given shape is proportional to the incoming velocity a (D'Alembert, [1749]: §42–43, 1752: §39).

[32] D'Alembert, 1752: xxxi. ...la pression du filet de Fluide qui glisse immédiatement sur la surface du corps. Pour cela il est nécessaire de connoître la vitesse des particules de ce filet.

[33] Cf. Truesdell, 1954: LIII, Grimberg, 1998: 44–46, Darrigol, 2005: 20–21.

[34] D'Alembert, [1749]: §39, 1752: §36.

[35] D'Alembert, [1749]: §40, 1752: §37. This vanishing, as we know, characterizes an ideal fluid; d'Alembert did not relate it to the nature of the fluid.

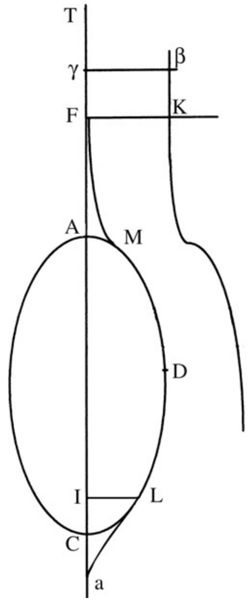

Fig. 4. Figure 14 of D'Alembert, [1749] redrawn. Not all elements shown here are used in our arguments.

a head–tail symmetry,[36] the contributions to the drag from two symmetrically located points would be equal and of opposite sign and thus cancel.[37] In order to avoid the vanishing of the drag, he assumed that the attachment point M and the separation point L are not symmetrically located:

From there it follows that the arcs LD and DM cannot be equal; because, if they were, the quantity—$\int 2\pi y \mathrm{d}y(p^2+q^2)$ would be equal to zero so that the body would not experience any force from the fluid: which is contrary to experiments.[38]

This stress on "experiments", already present in the 1749 manuscript and which will not reappear in d'Alembert's 1768 paradox paper, seems to reflect just common sense. It cannot be explained by d'Alembert's hypothetical desire to adhere to late recommendations by the Berlin Academy which emphasized comparisons with experiments for the 1750 prize on resistance of fluids. D'Alembert did not seem pleased with such late changes and these recommendations were probably formulated only in May 1750.[39]

D'Alembert's new idea, compared to Euler, is to consider the drag as the resultant of the pressure forces directed along the normal to the surface of the body over its entirety. But for d'Alembert it is still unimaginable to obtain a vanishing drag.

5. Euler and the 'Dilucidationes' (1756)

The *Dilucidationes de resistentia fluidorum* (Enlightenment regarding the resistance of fluids) have been written in 1756, one year after Euler established his famous equations in their final form.[40] In his review of previous efforts to understand the drag problem for incompressible fluids, Saint-Venant[41] writes the following about the *Dilucidationes*:

And it is obvious that, when the flow is assumed indefinite or very broad, the theory of the *Dilucidationes* can only be and actually is just a return to the vulgar theory, ... [42]

Here, Saint-Venant understands by "vulgar theory" the impact theory which goes back to the seventeenth century. Actually, in 1756 Euler was rather pessimistic regarding the applicability of his equations to the drag problem:

But the results which I have presented in several previous memoirs on the motion of fluids do not help much here. Because, even though I have succeeded in reducing everything that concerns the motion of fluids to analytical equations, the analysis has not reached the sufficient degree of completion which is necessary for the solving of such equations.[43]

Truesdell discusses the *Dilucidationes* in detail.[44] Actually this paper is quite famous because of a remark Euler made on the cavitation that arises from negative pressure in incompressible fluids. Truesdell is also rightly impressed by Euler's success in doing something non-trivial with his equation for flows around a parabolic cylinder; for this Euler uses a system of curvilinear coordinates based on the streamlines and their orthogonal trajectories.

The *Dilucidationes* are however not contributing much to our understanding of drag. In Section 15, Euler expresses his doubts regarding the applicability of his 1745 calculation to both the front and the back of a body (which would result in vanishing drag):

...the boat would be slowed down at the prow as much as it would be pushed at the poop ....[45]

We must mention here that, because of a possible non-vanishing transfer of kinetic energy to infinity, the modern theory of the d'Alembert paradox does not apply to flow with a free surface, such as a boat on the sea.

Thus, in the *Dilucidationes* we find a first attempt to introduce a new analytical treatment of streamlines unrelated to the previous theories and coming closer to the modern description of a fluid flow. Nevertheless, Euler does not succeed in using his 1755 equations to improve our understanding of the drag problem.

36 In d'Alembert [1752] this additional symmetry is explicitly assumed; in d'Alembert, [1749] the language used only suggests such a symmetry.

37 D'Alembert, [1749]: §62, 1752: §70.

38 D'Alembert, 1752: §70. Delà il s'ensuit que les arcs LD, DM ne sauroient être égaux ; car s'ils l'étoient, alors la quantité $-\int 2\pi y \mathrm{d}y(p^2+q^2)$ seroit égale à zéro de manière que le corps ne souffriroit aucune pression de la part du fluide : ce qui est contre l'expérience.

39 D'Alembert, 1752: xxxviii; Yushkevich and Taton, 1980: 312–314; Grimberg, 1998: 9.

40 Euler, 1755, 1756.

41 Saint-Venant, [1888], probably mostly written around 1846.

42 Saint-Venant, [1888]: 35. Et il est évident que, lorsque le courant est supposé indéfini ou très large, la théorie des *Dilucidationes* d'Euler ne peut être et n'est réellement qu'un retour pur et simple à la théorie vulgaire,

43 Euler, 1760: 200. Quae ego etiam nuper in aliquot dissertationibus de motu fluidorum exposui, nullum subsidium huc afferunt. Etiamsi enim omniam quae ad motum fluidorum pertinent, ad aequationes analyticas reduxi, tamen ipsa Analysis minime adhuc ita est exculta, ut illis aequationibus resoluendis sufficiat.

44 Truesdell, 1954: C–CVII.

45 Euler, 1760: 206 ...puppis nauis paecise tanta vi propelleretur, quanta prora repellitur....

6. Borda's memoir (1766)

In his memoir Borda, a prominent French "Geometer" and experimentalist, studies the loss of "live force" (energy) in incompressible flows, in particular in pipes whose section is abruptly enlarged.[46] At the end of his memoir Borda gives an example of what would be, in his opinion, "a bad use" of the principle of conservation of live forces. This is precisely the problem of determining the drag force that a moving fluid exerts upon a body at rest. The particles of the fluid in the neighborhood of the body "delineate curved lines or rather move in small curved channels"; the pressure force acting upon the body has to be determined. But the channels become narrower at certain locations similarly to a siphon, so that the principle of live forces cannot be used. To prove this point he then presents the following argument for the vanishing of the drag:

> . . . suppose that the body D moves uniformly through a quiescent fluid, driven by the action of the weight P. According to this principle [of live forces], the difference of the live force of the fluid must be equal to the difference of the actual descent of the weight; however, since the motion is supposed to have reached uniformity, the difference of the live forces equals zero. Therefore, the difference of the actual descent is also zero, which cannot happen unless the weight P is itself zero. As the weight P measures the resistance of the fluid, the supposition of the principle [of live forces] necessarily leads to a vanishing resistance.[47]

This constitutes the first derivation of the d'Alembert paradox using an energy dissipation argument. Borda's explanation of why the live-force conservation argument is inapplicable rests on the aforementioned analogy with the siphon problem. This is illustrated by a figure[48] not reproduced here because of its poor quality. There one sees a fillet of fluid narrowing somewhat as it approaches the body. The modern concept of dissipation in high-Reynolds-number flow being confined to regions with very strong velocity gradients is definitely not what Borda had in mind.

Borda's reasoning is correct, but like Euler in 1745 and d'Alembert in 1749, he does not formulate the vanishing of the drag as a paradox. In his remarks Borda addresses neither the question of the nature of the fluid, nor the consequences of having stationary streamlines, nor the problem of the contact between the fluid and the body (absence of viscosity in the case of ideal flow) which, as we know, are quite central to the understanding of the paradox.

7. D'Alembert's memoirs on the paradox (1768 and 1780)

In Volume V of his "Opuscules" published in 1768, a part of a memoir is entitled "Paradox on the resistance of fluids proposed to geometers."[49] D'Alembert considers again an axisymmetric body, but now with a head–tail symmetry. More precisely, he assumes a plane of symmetry perpendicular to the direction of the incompressible flow at large distance and dividing the body into two mirror-symmetric pieces. To avoid the problem of possible separation of streamlines upstream and downstream of the body, he assumes that the front part and the rear part of the body have needle-like endings. First of all he asserts that the velocities at every location in the fluid are perfectly symmetric in front/rear of the body, and that

> . . . under this assumption the law of the equilibrium and the incompressibility of the fluid will be perfectly obeyed, because, the rear part of the body being similar and equal to its front part, it is easy to see that the same values of p and q [i.e. the velocity components] which will give at the first instant the equilibrium and incompressibility of the fluid at the front part will give the same results for the rear part. [50]

This statement is directly related to the remark in Section 70 of d'Alembert's 1752 treatise. In fact, the assumption used by d'Alembert in 1749 and 1752 to avoid a paradox is here lifted, since no separation of streamlines occurs except at the needle-like end points. D'Alembert here assumes that the solution with mirror symmetry is the only one: "The fluid has only one way to be moved by the encounter of the body." The pressure forces at the front and rear part of the body are then also axisymmetric and mirror symmetric. Hence they combine into a force of resistance (drag) which vanishes. D'Alembert concluded:

> Thus I do not see, I admit, how one can satisfactorily explain by theory the resistance of fluids. On the contrary, it seems to me that the theory, developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance; a singular paradox which I leave to future Geometers to elucidate. [51]

It is clear that d'Alembert's argument is less general than that of Borda, since he is restricting the formulation of the paradox to bodies with a head–tail symmetry. Nevertheless, d'Alembert is the first one to seriously propose the vanishing of the drag as a paradox. Twelve years later in Volume VIII of his "Opuscules" d'Alembert revisits the paradox in the light of a letter received from "a very great Geometer" who is not named and who points out that, when considering the flow inside or around a

46 Borda, 1766.

47 Borda, 1766: 604–605. . . . supposons que le corps D se meuve uniformément dans un fluide tranquille, entraîné par l'action du poids P: on sait que suivant le principe, la différence de la force vive du fluide devra être égale à la descente actuelle du poids P; mais puisque le mouvement est censé parvenu à l'uniformité, la différence des forces vives $= 0$; donc la différence de la descente actuelle sera aussi $= 0$, ce qui ne se peut pas à moins que le poids P ne soit lui-même $= 0$: or le poids P marque la résistance du fluide : donc la supposition du principe dont il s'agit, donne toujours une résistance nulle.

48 Borda, 1766: Figure 14, found at the end of the 1766 volume on p. 847.

49 D'Alembert, 1768. In the eighteenth century "Geometer" was frequently used to mean "mathematician" (pure or applied).

50 D'Alembert, 1768: 133. . . . dans cette supposition les loix de l'équilibre & de l'incompressibilité du fluide seront parfaitement observées; car la partie postérieure étant (*hyp.*) semblable et égale à la partie antérieure, il est aisé de voir que les mêmes valeurs de p & de q; qui donneront au premier instant l'équilibre & l'incompressibilité du fluide à la partie antérieure, donneront les mêmes résultats à la partie postérieure.

51 D'Alembert, 1768: 138. Je ne vois donc pas, je l'avoue, comment on peut expliquer par la théorie, d'une maniere satisfaisante, la résistance des fluides. Il me paroît au contraire que cette théorie, traitée & approfondie avec toute la rigueur possible, donne, au moins en plusieurs cas, la résistance absolument nulle ; paradoxe singulier que je laisse à éclaircir aux Géometres [*sic*].

symmetric body, there may be, in addition to the symmetric solution, another one which does not possess such symmetry and to which d'Alembert's argument for the vanishing of the resistance does not apply.[52] D'Alembert concurs and discussed the issue at length. It should however be noted that a breaking of the symmetry was already assumed by him in his early work on the resistance when he assumed that the (hypothetical) points of attachment and detachment of the streamline following the body are not symmetrically located (see Section 4).

Thus d'Alembert was definitely the first to formulate the vanishing of the drag as a paradox within the accepted model of that time, namely incompressible fluid flow, implicitly taken as ideal.[53] He was however formulating it only for bodies with head–tail symmetry, not realizing that techniques introduced by Euler and Borda could have allowed him to obtain the paradox for bodies of arbitrary shapes.

8. Saint-Venant and the first precise formulation of the paradox (1846)

In three notes published in 1846 and then in a memoir published in 1847, Saint-Venant gives for the first time a general formulation of the paradox. A detailed write-up, mostly dating from the same period, was published only posthumously in 1888 and contains also a very interesting discussion of previous work.[54] Saint-Venant's memoir marks the beginning of the modern theory of the d'Alembert paradox which was to flourish, in particular with major contributions by Ludwig Prandtl.[55]

We here give only a very brief description of the key results of Saint-Venant. He first specified the properties of the incompressible fluid: the pressure force is normal to the surface element on which it is acting and therefore equal in all directions. The fluid moves steadily around a body at rest. He gives a derivation of the paradox, closely related to Borda's. Indeed, it suffices to establish the equation for the live forces acquired by the fluid to see that the live-force (energy) loss of the system is zero:

> If the motion has reached, as one always assumes, a steady state, the live force acquired by the system at every instant is zero; the work performed by the exterior pressures is also zero and the same applies to the work of the interior actions of the fluid whose density is assumed to be unchanging. Thus, the work of the impulse of the fluid on the body, and, consequently, the impulse itself, is necessarily equal to zero. [56]

He adds that the situation is different for a real fluid made of molecules in which there is friction at the contact between two neighboring fluid elements:

> But one finds another result if, instead of an ideal fluid – object of the calculations of the geometers of the last century – one uses a real fluid, composed of a finite number of molecules and exerting in its state of motion unequal pressure forces or forces having components tangential to the surface elements through which they act; components to which we refer as the friction of the fluid, a name which has been given to them since Descartes and Newton until Venturi.[57]

Thus, d'Alembert's paradox is explained by Saint-Venant for the first time as a consequence of ignoring viscous forces. Of course, a precise formulation of the paradox would not have been possible without a clear distinction between ideal and viscous fluids.

9. Conclusion

The problem of the resistance of bodies moving in fluids was – and still is – of great practical importance. It was thus naturally one of the first non-trivial problems tackled within the nascent eighteenth century hydrodynamics. Euler, who was not only a great "Geometer" but a person acutely aware of the needs of gunnery and ship building, tried – as we have seen – reaching beyond the old impact theory of Newton—and failed. He was lacking both the concept of viscous forces and a deep understanding of the global aspects of the topology of the flow around a body. His "failure" – as is frequently the case with major scientists – was however very creative: born was the idea of analyzing a steady flow into a set of fluid fillets of infinitesimal and non-uniform section; he also managed to calculate the forces acting on such fillet several years before there was any representation of the dynamics in terms of partial differential equations. Borda, being both a Geometer and an experimentalist, felt compelled to qualify as non-sensical a very simple live-force argument discovered by himself and which predicted a vanishing drag for bodies of arbitrary shape. D'Alembert, another brilliant Geometer, was probably less constrained by experimental considerations, and dared eventually to present the paradox known by his name. His proof reveals a very good understanding of the global topology of the flow but otherwise is very simple and limited intrinsically to bodies with a head–tail symmetry.

We must stress that the statement as a paradox is very much tied to the type of analytical representation of an ideal flow. From this point of view, experiments on flow past bodies, be they real or thought experiments, have rather been an obstacle to grasping the distinction between an ideal fluid and a real one. The same kind of epistemological obstacle has

[52] D'Alembert, 1780: 212; Birkhoff, 1950: 21–22.

[53] The idea of viscosity ripened only in the XIXth century, see e.g. Darrigol, 2005; in the eighteenth century there was only a concept of tenaciousness, e.g. resistance to the introduction of a body into fluid, which was still a long way from actual viscosity.

[54] Saint-Venant, 1846, 1847, [1888].

[55] Cf., e.g. Darrigol, 2005: Chap. 7.

[56] Saint-Venant, 1847: 243–244. Si le mouvement est arrivé, comme on le suppose toujours, à l'état de permanence, la force vive, acquise à chaque instant par le système, est nulle ; le travail des pressions extérieures est nul aussi, et il en est de même du travail des actions intérieures du fluide dont nous supposons que la densité ne change pas. Donc le travail de l'impulsion du fluide sur le corps, et, par conséquent, cette impulsion elle-même, est nécessairement zéro.

[57] Saint-Venant, 1847: 244. Mais on trouve un autre résultat si, au lieu du fluide idéal, objet des calculs des géomètres du siècle dernier, on remet un fluide réel, composé de molécules en nombre fini, et exercant dans l'état du mouvement, des pressions inégales ou qui ont des composantes tangentielles aux faces à travers desquelles elles agissent; composantes que nous désignons par le nom de frottement du fluide, qui leur a été donné depuis Descartes et Newton jusqu'à Venturi.

accompanied the earlier birth of the principle of inertia, which no experiment could at that time truly reveal; it was necessary to distance oneself from real conditions and to find an appropriate mathematical representation. Finding such representations for fluid dynamics was a painfully slow process: a full century elapsed between Euler's fragmentary results on drag and Saint-Venant's full understanding of the d'Alembert paradox.

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