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Formation Fronts of a Nonlinear Elastic Medium from a Medium without Shear Stresses

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Abstract—The fronts of phase transition of a medium without shear stresses to a nonlinear incompressible anisotropic elastic medium are considered. The mass flux through unit area of a front is assumed to be known. The variation of the tangential components of the medium's velocity and the variation of the arising shear stresses are studied. An explicit form of boundary conditions is found using the existence condition of a discontinuity front structure. The Kelvin–Voight viscoelastic model is adopted for this structure.

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INTRODUCTION

The formation fronts of an elastic incompressible weakly nonlinear anisotropic medium from a medium without shear stresses are considered. The medium ahead of the front is either a flux of noninteracting particles or an ideal fluid solidifying in the process of polymerization or cooling. Nonzero shear stresses may appear in the medium under formation.

Such fronts are of interest from the theoretical point of view, since the relations resulting from the conservation laws are not enough to describe them. The evolution analysis [1] of discontinuities requires additional conditions whose number is variable depending on the discontinuity rate and on the small perturbations of velocities at each side of the discontinuity. As shown in [2–4], a necessary number of additional relations can be obtained if we require the existence of discontinuity structures. As examples of discontinuities requiring additional relations, we can mention the combustion fronts [5], the ionization and recombination fronts [6], the nonclassical discontinuities in elastic media with small-scale dispersion and dissipation [7], and the discontinuity surfaces in elastoplastic media [8, 9]. Various fronts arising during the compaction of a flux of noninteracting particles with the formation of an isotropic elastic medium are considered in [10]. The formation fronts of an anisotropic elastic medium with a specific nonlinearity are discussed in [11]. In this paper we study the formation fronts of a nonlinear anisotropic medium with another type nonlinearity. Below we show that the nonlinearity type qualitatively changes the properties of the discontinuities under study.

The formation process of an elastic medium (in other words, the solidification process) is represented in the form of propagation of a plane front. The velocity of its motion in the direction orthogonal to its plane is assumed to be known; this velocity is specified by various processes dependent on the problems under consideration. For example, when an elastic medium is formed as a result of particle adhesion, the velocity of the front is determined on the basis of the mass conservation law with a known flux of adhering particles, whereas in the case of freezing or polymerization this velocity is determined on the basis of solving a heat problem or a problem of polymerization kinetics. In what follows, the velocity of a solidification wave is assumed to be constant and to be arbitrarily given, and the subject of study is the variation of the transverse velocity components and the stress components in the resulting elastic medium.

THE MOTION PROCESS IN A MEDIUM BEHIND A SOLIDIFICATION FRONT

It is assumed that an incompressible elastic medium is formed because of the propagation of a solidification front. The motion of the medium is considered in a coordinate system associated with this front. The x-axis of this coordinate system is directed along the normal to the front in the direction of its motion with respect to the medium, whereas the x_1-axis and the x_2-axis are parallel to the plane of the front. Since the medium is incompressible, the velocity component normal to the front is the same in the entire domain of elasticity: $v_x = -W$; only the transverse components of the velocity, stresses, and strains may be varied.

The system of differential equations for the transverse components is of the following form in the Lagrangian–Cartesian coordinate system:

$$\rho\,\frac{\partial v_\alpha}{\partial t} = \frac{\partial \sigma_\alpha}{\partial x}, \qquad \frac{\partial u_\alpha}{\partial t} = \frac{\partial v_\alpha}{\partial x}; \tag{1}$$

$$\sigma_\alpha = \frac{\partial \Phi(u_1, u_2)}{\partial u_\alpha}. \tag{2}$$

Here $\alpha = 1, 2$; ρ is the density of the formed medium; σ_α are the shear stresses in the elastic medium; and v_α are the tangential velocity components in this medium. In addition, we have $v_\alpha = \partial w_\alpha/\partial t$ and $u_\alpha = \partial w_\alpha/\partial x$, where w_α are the transverse displacements of the medium's points and u_α are the shear strains. The function Φ is the elastic energy of Lagrangian unit volume.

It is assumed that the shear strains u_α are small and the anisotropy of the formed medium is also small. Hence, we can represent the function Φ as [3]

$$\Phi(u_1, u_2) = \frac{f}{2}\,(u_1^2 + u_2^2) + \frac{g}{2}\,(u_2^2 - u_1^2) - \frac{\varkappa}{4}\,(u_1^2 + u_2^2)^2, \tag{3}$$

where f, g, and $\varkappa$ are the elastic parameters of the medium. The coefficient g characterizing the anisotropy in the plane of the front is assumed to be small. Hence, we can assume that the two last terms of (3) are of the same order of smallness. The appropriate numeration of the x_1- and x_2-axes allows one to make the sign of g to be positive.

The nonlinearity coefficient $\varkappa$ may be of an arbitrary sign. The media with different signs of $\varkappa$ behave differently [3] and should be considered individually. The case $\varkappa > 0$ is typical for metals and is discussed in [11]. In this paper the coefficient $\varkappa$ is assumed to be negative, which is typical for the rubber-type materials.

The system expressed by (1) is hyperbolic and has two families of characteristics. The characteristic velocities $\pm c_1$ correspond to the family of slow waves, whereas the characteristic velocities $\pm c_2$ correspond to the family of fast waves: $c_1 \leqslant c_2$. Note that c_1^2 and c_2^2 are the eigenvalues of the matrix $\frac{1}{\rho}\,\partial^2\Phi/\partial u_\alpha \partial u_\beta$. It is assumed that these eigenvalues are positive. In addition, c_1 and c_2 are dependent on u_1 and u_2. In the stress-free state, we have $c_{1,2}(0,0) = \sqrt{(f \mp g)/\rho} = c_{1,2}^0$.

The mass conservation law and the momentum conservation law on the solidification front are the main boundary conditions for system (1). If the solidification front is considered as a surface without thickness, then the mass conservation law specifies the velocity W of the front's motion on the basis of the given parameters for the incident flow of particles. Based on the transverse momentum conservation law, the following relations are obtained on the discontinuity:

$$\sigma_\alpha^+ = \rho\,W(V_\alpha - v_\alpha^+), \qquad \alpha = 1, 2. \tag{4}$$

Here σ_α^+ are the shear stresses behind the discontinuity, v_α^+ are the tangential velocity components behind the discontinuity, V_α are the tangential velocity components ahead of the discontinuity, ρW is the mass flux through the discontinuity, and ρ is the density of the formed medium. In (4) we take into account that there are no shear stresses in the medium ahead of the discontinuity.

Since the velocity W may be less or greater than the characteristic velocities of the elastic medium, the above conservation laws may be not enough to correctly formulate the necessary boundary conditions on the front and, then, it is necessary to formulate some additional boundary conditions.

This can be done if we require the existence of a discontinuity structure, which allows us to formulate the boundary conditions necessary to describe the evolutionary behavior of the discontinuity [2–4]. As a result, we are able to correctly formulate the problem devoted to the interaction between the discontinuity and the motion of the medium.

THE STRUCTURE OF THE SOLIDIFICATION FRONT

By the structure of a discontinuity front we mean a narrow zone where the process of phase transition is observed. This process is accompanied by the dissipative mechanisms and by the appearance of shear stresses and strains. In our case the process of phase transition consists in the solidification of the flux and in the formation of an elastic medium. In order to describe the above structure, we use the Kelvin–Voight viscoelastic model where there exist the elastic stresses (2) specified by the elastic potential and the viscous stresses characterized by the viscosity coefficient μ. The substantial variation of quantities proceeds in a layer whose thickness is dependent on μ.

For this model the system of equations for the transverse velocity components and the transverse strain components is of the following form expressed in terms of Lagrangian variables:

$$\rho\,\frac{\partial v_\alpha}{\partial t} = \frac{\partial \sigma_\alpha}{\partial x} + \mu\,\frac{\partial^2 v_\alpha}{\partial x^2}, \qquad \frac{\partial u_\alpha}{\partial t} = \frac{\partial v_\alpha}{\partial x}; \tag{5}$$

$$\sigma_\alpha = \frac{\partial \Phi(u_1, u_2)}{\partial u_\alpha}. \tag{6}$$

Here $\alpha = 1, 2$.

The leading boundary of the solidification zone is adjacent to the domain where $\sigma_\alpha = 0$; the effect of viscous terms disappears at infinity and the medium becomes elastic.

We seek a solution to this system in the form of a traveling wave whose velocity is equal to W. This solution is stationary in the coordinate system associated with the discontinuity front. Hence, we can introduce the Eulerian variable $\xi = -x + Wt$, where W is the velocity of the formed incompressible Kelvin–Voight medium with respect to the front, and we can to obtain a system of ordinary differential equations from Eqs. (5) and (6). Integrating the equations of motion with respect to ξ and eliminating v_α, we rewrite this system in the form

$$\mu W\,\frac{du_\alpha}{d\xi} = -\frac{\partial F(u_1, u_2)}{\partial u_\alpha}, \qquad \alpha = 1, 2. \tag{7}$$

Here the function $F(u_1, u_2)$ differs from the elastic potential $\Phi(u_1, u_2)$ given by (3) by the additional terms containing the front velocity W and by the integration constants Q_1 and Q_2:

$$F(u_1, u_2) = \frac{f - \rho W^2}{2}\,(u_1^2 + u_2^2) + \frac{g}{2}\,(u_2^2 - u_1^2) - \frac{\varkappa}{4}\,(u_1^2 + u_2^2)^2 + Q_1 u_1 + Q_2 u_2. \tag{8}$$

Recall that in this paper we consider the media with $\varkappa < 0$.

The variation of quantities along the integral curves of (7) can be considered as the motion of the point (u_1, u_2) along the slope of the relief defined by the function $F(u_1, u_2)$ on the plane u_1, u_2. The velocity of this motion is specified by the gradient of this function.

In order to find the solution describing the structure of the discontinuity front, we formulate the following boundary conditions. At the leading boundary (i.e., on the plane $\xi = 0$), we put $u_1 = 0$ and $u_2 = 0$ for $\xi = 0$, which means that the strains are absent when the Kelvin–Voight medium is formed. At the other boundary ($\xi = \infty$), the derivatives are equal to zero: $du_\alpha/d\xi = 0$, which means that the solution asymptotically corresponds to the parameters u_α^+ and σ_α^+ of the elastic medium behind the front. These conditions allow us to express the integration constants Q_1 and Q_2 in terms of the flow parameters u_α^+ for $\xi = \infty$. For given f, g, $\varkappa$, ρ, and μ, the sought solution is specified by the parameters W, Q_1, and Q_2. Varying the parameters W, Q_1, and Q_2, we obtain the set u_α^+ of states; this set is an analog of the shock adiabat for the solidification front. The construction of the shock adiabat is the most important part of conclusions made on the basis of the analysis of the above structure.

Thus, our analysis of the discontinuity structure is reduced to finding the integral curve described by system (7) on the plane u_1, u_2. This curve leaves the coordinate origin and arrives at one of the singular points of this system. The singular points of (7) are the stationary points of the function $F(u_1, u_2)$: $\partial F/\partial u_1 = 0$ and $\partial F/\partial u_2 = 0$.

The type of each singular point is specified by the signs of the eigenvalues λ of the matrix formed by the second derivatives of the function $F(u_1, u_2)$ at this point. According to [12], near a singular point we represent the solution to (7) in the form $q_\alpha = \hat{q}_\alpha e^{\lambda\xi}$, $q_\alpha = u_\alpha - u_\alpha^+$. Hence, the eigenvalues λ are the roots of the determinant

$$\left|\frac{\partial^2 F}{\partial u_\alpha \partial u_\beta} + \lambda\,\mu\,W\delta_{\alpha\beta}\right| = \left|\frac{\partial^2 \Phi}{\partial u_\alpha \partial u_\beta} - (\rho W^2 - \lambda\mu W)\delta_{\alpha\beta}\right| = 0.$$

Taking into account (8) and the representation of the characteristic velocities in terms of ρ and $\Phi(u_1, u_2)$, we get

$$\lambda_{1,2} = \frac{\rho}{\mu W}\,[W^2 - c_{1,2}^2(u_1, u_2)].$$

If $0 < W < c_1(u_1^+, u_2^+)$, then the singular point $u_\alpha = u_\alpha^+$ is a node with incoming integral curves; if $c_1(u_1^+, u_2^+) < W < c_2(u_1^+, u_2^+)$, then this singular point is a saddle point; and if $W > c_2(u_1^+, u_2^+)$, then this singular point is a node with outgoing integral curves. The type of a singular point may be changed by varying the parameters in the case of merging with other singular points or in the case of splitting into two or more singular points.

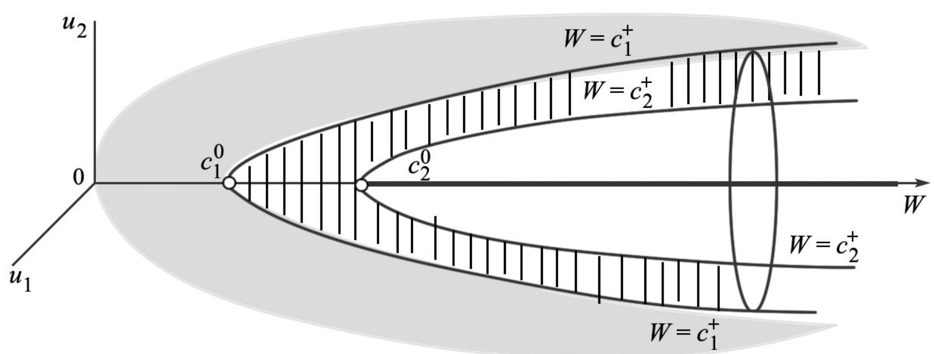
ANALYSIS OF THE STRUCTURE AND DISCONTINUITY TYPES

For various values of the velocity W, here we discuss the existence of an integral curve of (7) such that this curve leaves the coordinate origin on the plane u_1, u_2 and arrives at a singular point of (7).

1. The case when $0 < W < c_1(0,0)$, i.e., when $\rho W^2 < f - g$.

Let $Q_1 = 0$ and $Q_2 = 0$. A singular point of the function $F(u_1, u_2)$ is situated at the coordinate origin. If $0 < W < c_1(0,0)$, then this point is a minimum point ensured by the quadratic terms in the expression of the function $F(u_1, u_2)$. The fourth-degree term induces a positive definite matrix of second derivatives for all $u_1 \neq 0$ and $u_2 \neq 0$. For $0 < W < c_1(0,0)$, $Q_1 = 0$, and $Q_2 = 0$, thus, there exists a unique stationary point; this point is an incoming node of system (7).

If one of the values of Q_1 and Q_2 is not equal to zero, then this point is displaced and remains to be unique. In this case there exists an integral curve such that this curve leaves the coordinate origin and arrives at this point characterizing the state behind the solidification front: $u_\alpha = u_\alpha^+$, $\alpha = 1, 2$. Since this point is an incoming node, from the above discussion it follows that $W < c_1(u_1^+, u_2^+)$. Such fronts are said to be slow. Varying Q_1 and Q_2 for a given $W < c_1(0,0)$, we can move the minimum of $F(u_1, u_2)$ at any point of the plane u_1, u_2. Varying W within the above range, we conclude that, in the phase space u_1, u_2, W, the layer bounded on the left by the condition $W > 0$ and bounded on the right by the condition $W = c_1(0,0) = c_1^0$ corresponds to the region of slow solidification fronts. In the figure, this layer is highlighted in gray.



Shock adiabat of the solidification front.

Our analysis of the existence and evolution behavior of such fronts requires no additional conditions: it is sufficient to use only the conditions that follow from the conservation laws.

2. The case when $c_1(0,0) < W < c_2(0,0)$, i.e., when $f - g < \rho W^2 < f + g$.

If $W = \text{const}$ and $Q_1 = Q_2 = 0$, then, on the plane u_1, u_2, the coordinate origin is a saddle point of the function $F(u_1, u_2)$; this point can also be considered as a stationary point of this function for $\partial^2 F/\partial u_1^2 < 0$, $\partial^2 F/\partial u_2^2 > 0$, and $\partial^2 F/\partial u_1 \partial u_2 = 0$) as well as a singular point of (7). In addition, the function $F(u_1, u_2)$ has two minimum points; these points are symmetric on the u_1-axis. This means that there exist integral curves such that these curves leave the coordinate origin and arrive at these points. If we increase the absolute value of Q_2 in such a way that $Q_2 < 0$ and $Q_1 = 0$, then the saddle point and these minima begin to be displaced.

Their positions are defined by the equations

$$\frac{\partial F(u_1,u_2)}{\partial u_1} = (f-g-\rho W^2)u_1 - \varkappa(u_1^2+u_2^2)u_1 = 0,$$
$$\frac{\partial F(u_1,u_2)}{\partial u_2} = (f+g-\rho W^2)u_2 - \varkappa(u_1^2+u_2^2)u_2 = -Q_2.$$

From the first equation it follows that the saddle point moves to the right along the u_2-axis for $u_1 = 0$; from the second equation it follows that the saddle point moves up along the u_2-axis if $Q_2 < 0$ increases in its absolute value. The minimum points of the function $F(u_1, u_2)$ are situated on the circle

$$u_1^2 + u_2^2 = \frac{\rho W^2 - f + g}{-\varkappa} \tag{9}$$

and their distance from the u_1-axis increases with increasing $|Q_2|$. The radius

$$r_* = \sqrt{(f-g-\rho W^2)/\varkappa}$$

of this circle increases with increasing W.

The above singular points merge at the points where the circle intersects the u_2-axis. At these points the velocity W of the solidification front is coincident with the slow characteristic velocity $W = c_1(u_1^+, u_2^+) = c_1^+$ behind the front. With further increase in $|Q_2|$, there remains only the stationary point on the u_2-axis; this point is a minimum point. If $Q_2 > 0$, then the position pattern of stationary points and integral curves is symmetric to the above pattern with respect to the u_1-axis.

Let us consider the case when $Q_1 = 0$ and the values of $|Q_2|$ are moderate; in this case there exist three stationary points of F. Then, the only integral curve leaving the coordinate origin goes to the saddle point on the u_2-axis. The coordinates $u_1 = 0$ and $u_2 = u_2^+$ of this point may correspond to the conditions at $\xi = \infty$. The type of this singular point (saddle) indicates that the inequality $c_1^+ < W < c_2^+$ is valid behind the corresponding front; here c_1^+ and c_2^+ are the slow and fast characteristic velocities behind the front. Such fronts are said to be fast.

Now we consider the positions of the stationary points for $Q_1 < 0$. In this case the integral curves intersect the u_2-axis from left to right in the direction of increasing u_1. The integral curve leaving the coordinate origin follows the same direction. The saddle point moves from the u_2-axis to the left, whereas the integral curve leaving the coordinate origin is not able to arrive at this point. The minimum point of the function $F(u_1, u_2)$ belonging to the right half-plane for $Q_1 < 0$ moves to the right from its position for $Q_1 = 0$. The integral curve leaving the coordinate origin arrives at this point and, then, represents the state behind the structure of the slow front. The point representing the state behind the front is situated outside the circle expressed by (9); by choosing the appropriate values of Q_1 and Q_2, obviously, this point can be moved to any point on the plane u_1, u_2 with $W = \text{const}$, except for the interior of circle (9).

Let us discuss two discontinuities such that they have the same velocities (therefore, they can be considered as a single one) and are of the following structure: the first jump from the coordinate origin to the saddle point on the u_2-axis and the second jump to the minimum point on the circle expressed by (9). Using these discontinuities, it is possible to arrive at the points of circle (9) for $Q_1 = 0$. Such a united discontinuity should be considered as a slow solidification front according to the type of a finite singular point.

Thus, the set of possible states behind the fronts for a given W from the interval $c_1(0,0) < W < c_2(0,0)$ is the exterior of (9) on the plane u_1, u_2 (the slow fronts) and the interval of the u_2-axis belonging to interior of this circle.

Varying the front velocity W within the range $c_1(0,0) < W < c_2(0,0)$, we determine the set of states that can be observed behind the solidification front in the three-dimensional phase space u_1, u_2, W. This set contains the three-dimensional domain outside the rotation surface (9) intersecting the plane u_2, W along the lines where $W = c_1^+$. In the figure, this domain (highlighted in gray) is a continuation of the three-dimensional set of states containing the above slow fronts. In the range of W, there appears a two-dimensional set as a part of the plane $u_1 = 0$ inside the above rotation surface. The domain containing the states corresponding to the fast fronts is indicated in the figure by vertical hatching. Note that it is necessary to introduce the additional condition $u_1^+ = 0$ on the solidification front; this condition follows from the existence of a solution to the problem on a structure in this layer.

3. The case when $W > c_2(0,0)$, i.e., when $\rho W^2 > f + g$.

This case differs from the previous one by the fact that the second derivatives of the function $F(u_1, u_2)$ are negative at the coordinate origin. This fact means that, for small values of Q_1 and Q_2, there are no stationary points of $F(u_1, u_2)$ such that integral curves arrive at them near the coordinate origin. As in the previous case, the minima of $F(u_1, u_2)$ are situated on the plane u_1, u_2 outside the circle expressed by (9) for the corresponding values of Q_1 and Q_2 and there exists an integral curve leaving the coordinate origin to these points. Since these points are minima, this means that $W < c_2(u_1^+, u_2^+)$; in other words, these fronts are slow ones. The domain of possible values of u_1^+, u_2^+ is adjacent to the domain of the values found previously for the range $c_1(0,0) < W < c_2(0,0)$ and highlighted in the figure by shadowing. If a value of W is fixed, then in this range we can observe two segments of the u_2-axis. These segments consist of saddle points and are adjacent to circle (9) from its interior. On the u_2-axis, their inner boundaries are specified by the condition $\partial^2 F/\partial u_2^2 = 0$, or

$$u_2^2 = -\frac{1}{3\varkappa(\rho W^2 - f - g)} \qquad (\varkappa < 0). \tag{10}$$

At these points, we have $W = c_2(u_1^+, u_2^+) = c_2^+$. In the phase space u_1, u_2, W, thus, the set of states behind fast waves belongs to the plane u_2, W between the curves expressed by (9) and (10).

In addition to the solidification fronts with varying u_α, there may exist fronts such that the values of u_α remain to be equal to zero. In the previous ranges of W, these trivial discontinuities are included in the continuous series of slow or fast discontinuities. In the case $W > c_2(0,0)$, however, other fronts with small variations of u_1 and u_2 do not exist. The velocity of such a front exceeds the velocity of any small perturbations that cannot change this front. Such a front is said to be ultra-fast.

The additional conditions $u_1^+ = 0$ and $u_2^+ = 0$ should be adopted for the existence of the above fronts. The states behind such a front are illustrated in the figure by the one-dimensional set in the form of the segment on the W-axis to the right of the point $c_2(0,0)$ and are distinguished by the heavy line.

CONCLUSION

The problem of one-dimensional motion is considered when there exists the phase transition front where a medium without shear stresses is transformed to a nonlinear anisotropic incompressible elastic medium.

It is shown that, in the phase transition front, new boundary conditions should be formulated depending on the relations between the velocity of this front and the velocity of small perturbations. These new boundary conditions are additional to the basic boundary conditions obtained from the conservation laws. The new additional conditions are found as the existence conditions for the structure of discontinuities under the assumption that the formed medium possesses the elastic and viscous stresses (the Kelvin–Voight medium) whose effect specifies the structure of waves. The set of states behind the various fronts that move along a given state (the shock adiabat) is studied. In the space of states, this set contains the parts of different dimensions from one to three in accordance with the number of additional relations on the front.

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