ISSN 0015-4628, Fluid Dynamics, 2017, Vol. 52, No. 5, pp. 678–681. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © N.E. Leont'ev, A.V. Smikhovskii, 2017, published in Izvestiya Rossiiskoi Akademii Nauk, Mekhanika Zhidkosti i Gaza, 2017, No. 5, pp. 86–89.

Steady-State Axisymmetric Flows of an Incompressible Fluid through Rotating Porous Media with Regard to the Coriolis Force

N. E. Leont'ev* and A. V. Smikhovskii

Lomonosov Moscow State University, Faculty of Mechanics and Mathematics,
Leninskiye Gory 1, Moscow, 119991 Russia
*e-mail: *leontiev_n@mail.ru*

Received March 11, 2017

Abstract—Exact solutions to the problem of steady-state axisymmetric flow of an incompressible fluid through rotating rigid body with regard to the centrifugal and Coriolis forces are constructed. The case of the locally transversally isotropic porous skeleton and the quadratic resistance force in the law of flow is considered. Estimates of the practical applicability of the solutions obtained are given. An analysis of increase in the length of the trajectory of a liquid particle due to its deviation from the radial direction in the frame of reference connected with the skeleton is carried out. This is of interest for applications related to deep-bed filtration of suspensions.

Keywords: flow through a porous medium, nonlinear (non-Darcian) flows, Coriolis force.

DOI: 10.1134/S001546281705009X

Flows through rotating porous media are considered with regard to inertial forces, in particular, the Coriolis force. Similar problems may be of interest for describing both operation of apparatuses which use the action of centrifugal forces (centrifuges, separators, etc.) and flows in rotating porous particles.

Earlier, the exact and approximate solutions obtained with regard to the Coriolis force [1–4] were considered for flows through porous media with various linear laws of flow, in particular, the Darcy's and Brinkman laws were used. In the present study the exact solutions are obtained with regard to nonlinearity of the law of flow through a porous medium and anisotropy of the porous skeleton.

1. CONSTITUTIVE EQUATIONS

Steady-state flow of a homogeneous incompressible Newtonian fluid through a rigid porous body rotating with a constant angular velocity $\boldsymbol{\omega}$ with respect to a space-fixed axis can be described by the following equation of motion and the continuity equation written in the reference frame connected with the body

$$\begin{gathered} -\nabla(mp) + \rho m\mathbf{F} - m\mathbf{F}_{\text{res}} + \mu\Delta\mathbf{u} + \rho m\omega^2 r\mathbf{e}_r - 2\rho[\boldsymbol{\omega}\times\mathbf{u}] = 0, \\ \operatorname{div}\mathbf{u} = 0 \end{gathered} \tag{1.1}$$

where m is the porosity which, generally speaking, depends on the spatial coordinates, p is the pressure, ρ and μ are the constant density and viscosity of fluid, r is the distance from the axis of rotation, $\mathbf{u}$ is the Darcy velocity, $\mathbf{F}$ is the mass density of the external body forces, and $\mathbf{e}_r$ is the radial unit vector of the cylindrical coordinate system.

In the very simple case of the homogeneous isotropic medium, $\mathbf{F}_{\text{res}}$ (a part of the resistance force which enters into the first equation and depends algebraically on the Darcy velocity) can be adequately approximated by the quadratic dependence (the Forchheimer law [5])

$$\mathbf{F}_{\text{res}} = \frac{\mu}{k}\mathbf{u} + \frac{\delta}{\sqrt{k}}\rho|\mathbf{u}|\mathbf{u}, \tag{1.2}$$

where k is the constant permeability and δ is a dimensionless constant which depends on the structure of the porous medium.

For the anisotropic medium we take one of the possible expressions for the Cartesian components of the force (the Einstein rule on summation over the repeating indices is assumed to be fulfilled):

$$F_{\mathrm{res},i} = A_{ij}u_j + B_{ij}|\mathbf{u}|u_j; \qquad i,\ j = 1,\ 2,\ 3 \tag{1.3}$$

with the symmetric matrices A_{ij} and B_{ij}. This expression includes (1.2) as a particular case (the general form of the dependence was considered in [6, 7]). For the positive-definite matrices A and B the formula (1.3) ensures satisfaction of the second law of thermodynamics related to the positiveness of energy dissipation.

We note that the convection terms on the left-hand side of the first of the equations in (1.1) which take into account the inertiality of fluid in large-scale flows are omitted since the characteristic dimension of the problem under consideration is assumed to be much greater than the pore dimension. On the other hand, in (1.1) we retain the linear term with the Laplace operator (the Brinkman correction term) which arises in deriving the equation of motion in the theory of averaging even in the case of the anisotropic medium [8, 9]. This term can be significant in certain cases.

The inertia forces manifest themselves in appearance of the potential centrifugal force which can be eliminated from consideration using an appropriate redefinition of the pressure and the non-potential Coriolis force which does not do work; the formal validity of this (evident) representation was tested, for example, in [10].

2. AXISYMMETRIC STEADY-STATE FLOWS

We will consider plane-parallel axisymmetric flows, which are of interest from the point of view of practical applications, in the case of the porous medium with concentric layers (local transversal isotropy; the local axis of rotation is directed along with the direction of increase in the coordinate r). We will assume that the pressure p and the porosity m depend on only the radius r of the polar (cylindrical) coordinate system, the Darcy velocity takes the form which automatically satisfies the continuity equation:

$$\mathbf{u} \equiv u_r\mathbf{e}_r + u_\theta\mathbf{e}_\theta = \frac{Q}{2\pi r}\mathbf{e}_r + \frac{\Gamma(r)}{2\pi r}\mathbf{e}_\theta,$$

where Q is the conserving radial volume flow rate (per unit of length in the direction of the axis of rotation), $\mathbf{e}_\theta$ is the unit basis vector of the polar coordinate system directed toward the increase in the polar angle θ, and $\Gamma(r)$ is the circulation of the Darcy velocity along the circle of the radius r. There are no body forces.

By virtue of the adopted assumptions on the structure of the medium, in the local orthonormal physical basis $(\mathbf{e}_r,\ \mathbf{e}_\theta)$ the matrices in (1.3), which can be considered to be two-dimensional, will be diagonal with the nonzero components

$$A_{11} = a(r), \qquad A_{22} = b(r), \qquad B_{11} = c(r), \qquad B_{22} = d(r).$$

Neglecting the Brinkman correction term in (1.1), we obtain the following fourth-order algebraic equation for the quantity $\Gamma(r)$ which determines the circumferential component of the Darcy velocity ($\Gamma < 0$ when $Q > 0$ and $\omega > 0$)

$$\frac{d^2}{(2\pi r)^2}\Gamma^4 + \left(\frac{d^2Q^2}{(2\pi r)^2} - b^2\right)\Gamma^2 - 2\gamma Qb\cdot\Gamma - \gamma^2Q^2 = 0, \qquad \gamma = \frac{2\rho\omega}{m}. \tag{2.1}$$

This determines completely the field of the Darcy velocity from which we can find the pressure field

$$p(r) = \frac{\int F(r)\,dr + \mathrm{const}}{m(r)} + \frac{\rho\omega^2r^2}{2},$$
$$F(r) \equiv -mu_r\left(a + c\sqrt{u_r^2 + u_\theta^2}\right) + 2\rho\omega u_\theta.$$

We note that the solution obtained can be used both in the case of total filling of a porous sample by fluid (for example, for flow in the annular layer of a porous medium with supply of fluid across the inner or outer boundary) and in the case of quasi-steady-state flows with the free surface inside the porous medium (the instantaneous saturation model in the Kraiko terminology [11]).

For the sake of the more detailed analysis we will consider two particular cases.

3. LINEAR RESISTANCE LAW

In the first case of slow flows which obey Darcy's law ($c \equiv 0$ and $d \equiv 0$) we can find the Darcy velocity field in the explicit form:

$$\Gamma(r) = -\frac{\gamma Q}{b(r)}, \qquad \mathbf{u} = \frac{Q}{2\pi r}\left(\mathbf{e}_r - \frac{2\rho\omega k}{m\mu}\mathbf{e}_\theta\right),$$

while the pressure distribution is given by the equality

$$p(r) = \frac{1}{m(r)}\left(\text{const} - \int \frac{Q}{2\pi r}\left(m(r)\cdot a(r) + \frac{4\rho^2\omega^2}{m(r)b(r)}\right)dr\right) + \frac{\rho\omega^2 r^2}{2}.$$

In particular, for constant a, b, and m the pressure field takes the form:

$$p(r) = \frac{\rho\omega^2 r^2}{2} - \frac{Q}{2\pi}\left(a + \frac{4\rho^2\omega^2}{m^2 b}\right)\ln r + \text{const}.$$

In the last case the trajectories of liquid particles are logarithmic spirals [12]. We note that the pressure field does not need to be monotonic in r and, generally speaking, the flow is not potential.

In the case considered the physical applicability of the solution obtained is determined by smallness of the Reynolds number $\text{Re} = \rho u\sqrt{k}/\mu \lesssim 1$. This can be reached due to the choice of appropriate pressure difference (we assume that $r \geq r_0 > 0$, $r_0 = \text{const}$).

The significance of the effect of the Coriolis force is determined by the following condition on the Ekman number: $\text{Ek}^{-1} = \rho\omega k/(m\mu) \gtrsim 1$. The last condition can be satisfied in practice for media with large pores; for example, for water flow in a sample with the pores of the dimension $d_0 \sim \sqrt{k} \sim 1$ mm the radial and circumferential velocities are of the same order $u_\theta \sim u_r$ even for $\omega \sim 10^1$ rad/s.

An important conclusion is the possibility to increase considerably the path of a liquid particle in the rotating sample. This can be essential for a series of practical applications (deep-bed filtration of suspensions, polluted fluids, etc.).

As noted above, the general solution (2.1) was obtained without regard for the Brinkman correction term. In the considered case of the linear resistance law, taking this term into account in (1.1) leads to change in the behavior of the solution only in narrow boundary zones whose relative thickness (as compared with the dimension L of the flow domain) is of the order of $\varepsilon = \sqrt{k/(mL^2)} \ll 1$; therefore, the presence of the boundary layer has almost no effect on the trajectories of liquid particles.

4. QUADRATIC RESISTANCE LAW

In the second case of fast flows ($\text{Re} \gg 1$) we can neglect the linear term in (1.3) by setting $a \equiv 0$ and $b \equiv 0$. This gives

$$\Gamma(r) = -\frac{Q}{\sqrt{2}}\left\{\left[1 + \left(\frac{4\pi\gamma r}{Qd}\right)^2\right]^{1/2} - 1\right\}^{1/2}.$$

In passage to the limit as $Q \to \infty$ (for large flow rates) we have $\Gamma(r) \sim -4\pi\omega\rho/(md)\cdot r$ so that at fairly large r the circumferential component of the flow velocity becomes much higher than the radial component: $|u_\theta| \gg |u_r|$. The opposite passage to the limit as $Q \to 0$ has no physical meaning since we study the case of high velocities. For the isotropic medium the significant deviation of the liquid particle trajectory from the radial direction arises when

$$\frac{|u_\theta|}{|u_r|} = \frac{|\Gamma|}{|Q|} \sim \frac{\omega\sqrt{k}}{m\delta u_r} \gtrsim 1.$$

This can easily be implemented in practice. For example, for water in the medium with pores of the order of 1 mm the trajectories deviate from the radial direction at $u_r \sim 1$ cm/s when $\omega \sim 10^1$ rad/s (in this case $\mathrm{Re} \gg 1$).

Summary. It is shown theoretically that for flows of liquid media in rotating porous samples there can exist situations in which the influence of the Coriolis force is essential. In particular, taking this force into account can significantly affect the trajectories of liquid particles with respect to the porous skeleton. This is of importance for a series of practical applications.

The authors wish to thank A.N. Golubyatnikov for useful discussions.

The work was carried out with support from the Russian Foundation for Basic Research (projects Nos. 15-01-00361 and 17-01-00037).

REFERENCES

1. D.A. Nield, "Spin-up in a Saturated Porous Medium," Transport in Porous Media **4**, 495–497, DOI: 10.1007/BF00179532 (1989).
2. V.G. Zhukov and V.M. Chesnokov, "Two-Dimensional Problem of Centrifugal Filtration," Theoretical Foundations of Chemical Engineering **45**, No. 4, 386–393, DOI: 10.1134/S0040579511040166 (2011).
3. P. Vadasz, "Fluid Flow through Heterogeneous Porous Media in a Rotating Square Channel," Transport in Porous Media **12**, No. 1, 43–54, DOI: 10.1007/BF00616361 (1993).
4. P. Vadasz, "Flow and Thermal Convection in Rotating Porous Media," in: K. Vafai (Ed.), *Handbook of Porous Media* (1st ed., Marcel Dekker, Inc., N.Y., Basel, 2000), P. 395–439.
5. R.E. Collins, *Flow of Fluids through Porous Materials* (Reinhold Publishing Corp., N.Y., 1961; Mir, Moscow, 1964).
6. Yu.I. Sirotin, "Tensor Functions of a Polar and Axial Vector Compatible with Texture Geometry," Journal of Applied Mathematics and Mechanics **28**, No. 4, 804–816, DOI: 10.1016/0021-8928(64)90065-6 (1964).
7. V.F. Pleshakov and Yu.I. Sirotin, "Anisotropic Vector Functions of a Vector Argument," Journal of Applied Mathematics and Mechanics **30**, No. 2, 301–313, DOI: 10.1016/0021-8928(67)90178-5 (1967).
8. S. Whitaker, *The Method of Volume Averaging* (Theory and Applications of Transport in Porous Media. Vol. 13, Springer Science+Business Media, Dordrecht, 1999), DOI: 10.1007/978-94-017-3389-2.
9. N.E. Leont'ev, *Fundamentals of the Theory of Flow through Porous Media* (Publishing House of the Center of Applied Investigations under the Mechanics and Mathematics Department of Lomonosov Moscow State University, Moscow, 2009) [in Russian].
10. J.-L. Auriault, C. Geindreau, and P. Royer, "Coriolis Effects on Filtration Law in Rotating Porous Media," Transport in Porous Media **48**, 315–330, DOI: 10.1023/a:1015720529464 (2002).
11. A.N. Kraiko and A.A. Makhmudov, "Solution of the Two-Dimensional Unsteady Problem of Percolation into a Porous Soil within the Framework of the Instantaneous Saturation Model," Fluid Dynamics **24** (4), 574–581, DOI: 10.1007/BF01052420 (1989).
12. M. Vygodsky, *Mathematical Handbook. Higher Mathematics* (Mir, Moscow, 1971).