ISSN 0012-2661, Differential Equations, 2017, Vol. 53, No. 5, pp. 617–622. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © A.V. Aksenov, 2017, published in Differentsial'nye Uravneniya, 2017, Vol. 53, No. 5, pp. 624–629.

PARTIAL DIFFERENTIAL EQUATIONS

Fundamental Solution of the Elasticity Theory Equations in Displacements for a Transversely Isotropic Medium

A. V. Aksenov
Lomonosov Moscow State University, Moscow, 119992 Russia
e-mail: aksenov.av@gmail.com, aksenov@mech.math.msu.su

Received June 20, 2016

Abstract—We consider a linear fourth-order elliptic partial differential equation describing the displacements of a transversely isotropic linearly elastic medium. We find the symmetries of this equation and of the inhomogeneous equation with the delta function on the right-hand side. Based on the symmetries of the inhomogeneous equation, we construct an invariant fundamental solution in elementary functions.

DOI: 10.1134/S0012266117050056

INTRODUCTION. PRELIMINARY RESULTS

The system of equilibrium equations in displacements for a transversely isotropic linearly elastic medium was reduced in [1] to a system of three linear inhomogeneous equations for the three displacement vector components. There are canonical linear fourth-order partial differential equations associated with the homogeneous equations. These canonical equations generalize the biharmonic equation describing the displacements of an isotropic linearly elastic medium. To find the displacements of a transversely isotropic linearly elastic medium subjected to a given bulk force, one needs to know the fundamental solutions of the canonical equations.

Reductions of the system of 3D elasticity equations in displacements to systems of higher-order equations based on operators more suitable for a numerical-analytical study than the Lamé operator are known as representations of the solution of the problem of elasticity theory and form part of the classical theory. In particular, reduction to tetraharmonic equations can be found in [2].

Fundamental solutions of linear equations of mathematical physics are often invariant under the transformations admitted by the original equation [3, p. 272]. In the present paper, we construct a fundamental solution using an algorithm for finding fundamental solutions of linear partial differential equations suggested in [4, 5]. The algorithm is based on symmetries admitted by linear partial differential equations with the delta function on the right-hand side. Let us present the main result in [4, 5] needed for the subsequent exposition.

Consider a linear pth-order partial differential equation

$$Lu \equiv \sum_{\alpha=1}^{p} A_\alpha(x) D^\alpha u = 0, \qquad x \in \mathbb{R}^m. \tag{1}$$

Here we use the standard notation: $\alpha = (\alpha_1, \ldots, \alpha_m)$ is a multi-index with integer nonnegative components, $\alpha = \alpha_1 + \cdots + \alpha_m$, and

$$D^\alpha \equiv \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^m}\right)^{\alpha_m}.$$

The fundamental solutions of Eq. (1) are the solutions of the equation

$$Lu = \delta(x - x_0). \tag{2}$$

It was shown in [6] that if $p \geq 2$ and $m \geq 2$, then Eq. (1) can only admit symmetry operators of the form

$$Z = \sum_{i=1}^{m} \xi^i(x)\frac{\partial}{\partial x^i} + \eta(x,u)\frac{\partial}{\partial u}, \qquad \frac{\partial^2 \eta}{\partial u^2} = 0.$$

The main Lie algebra of symmetry operators of Eq. (1) treated as a vector space is the direct sum of two subalgebras, the subalgebra consisting of operators of the form

$$X = \sum_{i=1}^{m} \xi^i(x)\frac{\partial}{\partial x^i} + \zeta(x)u\frac{\partial}{\partial u} \tag{3}$$

and the infinite-dimensional subalgebra generated by the operators

$$X_\infty = \varphi(x)\frac{\partial}{\partial u}, \tag{4}$$

where $\varphi(x)$ is an arbitrary solution of Eq. (1). Note that the symmetry operators (4) are symmetry operators of Eq. (2). In what follows, we only consider symmetry operators of the form (3). By $\underset{p}{X}$ we denote the pth-order prolongation of the symmetry operator (3).

Proposition 1. *An infinitesimal operator of the form* (3) *is a symmetry operator of Eq.* (1) *if and only if there exists a function* $\lambda = \lambda(x)$ *satisfying the identity*

$$\underset{p}{X}(Lu) \equiv \lambda(x)\, Lu \tag{5}$$

for any function $u = u(x)$ *in the domain of Eq.* (1).

Theorem 1. *The Lie algebra of symmetry operators of Eq.* (2) *is a subalgebra of the Lie algebra of symmetry operators of Eq.* (1). *This subalgebra is defined by the relations*

$$\begin{aligned} &\xi^i(x_0) = 0, \qquad i = 1, \ldots, m, \\ &\lambda(x_0) + \sum_{i=1}^{m} \frac{\partial \xi^i(x_0)}{\partial x^i} = 0. \end{aligned} \tag{6}$$

Let us state an algorithm for finding fundamental solutions based on symmetries [4, 5].

1. Find the general form of a symmetry operator of Eq. (1) and of the corresponding function $\lambda(x)$ satisfying identity (5).

2. Derive the general form of a symmetry operator of Eq. (2) from the symmetry operator of Eq. (1) based on the constraints (6).

3. Construct invariant fundamental solutions with the use of symmetries of Eq. (2).

4. Obtain new fundamental solutions from known ones with the use of symmetries of Eq. (2) (production of solutions).

Remark 1. To find generalized invariant fundamental solutions, one should seek invariants in the class of distributions.

The construction of an invariant fundamental solution in elementary functions for the equation of a transversely isotropic linearly elastic medium is the main result of the present paper.

MAIN EQUATIONS

Consider the following linear fourth-order differential equations introduced in [1]:

$$\begin{aligned} L_1 u &\equiv u_{xxxx} + 2u_{xxyy} + u_{yyyy} + B_1(u_{xxzz} + u_{yyzz}) + B_2 u_{zzzz} = 0, \\ L_2 u &\equiv B_3(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) + B_4(u_{xxzz} + u_{yyzz}) + u_{zzzz} = 0. \end{aligned} \tag{7}$$

Here B_1, B_2, B_3, and B_4 are positive constants characterizing a linearly elastic medium. The fundamental solutions of Eqs. (7) are the solutions of the equations

$$L_1 u = \delta(x)\delta(y)\delta(z), \qquad L_2 u = \delta(x)\delta(y)\delta(z). \tag{8}$$

Let us show that there exist changes of variables reducing Eqs. (7) [and accordingly Eqs. (8)] to coinciding equations. To this end, we pass to the new variables

$$\bar{z} = \frac{z}{\sqrt[4]{B_2}}, \qquad \bar{u} = \sqrt[4]{B_2}u$$

in the equations corresponding to the differential operator L_1. If we denote $\bar{z}$ by z and $\bar{u}$ by u in the resulting equations, then the corresponding equations in (7) and (8) acquire the form

$$L_3 u \equiv u_{xxxx} + 2u_{xxyy} + u_{yyyy} + b(u_{xxzz} + u_{yyzz}) + u_{zzzz} = 0, \tag{9}$$
$$L_3 u = \delta(x)\delta(y)\delta(z). \tag{10}$$

Here $b = B_1/\sqrt{B_2}$. Likewise, one can make the change of variables

$$\bar{x} = \frac{x}{\sqrt[4]{B_3}}, \qquad \bar{y} = \frac{y}{\sqrt[4]{B_3}}, \qquad \bar{u} = \sqrt[4]{B_3}u$$

in the equations corresponding to the differential operator L_2 and denote $\bar{x}$ by x, $\bar{y}$ by y, and $\bar{u}$ by u in the resulting equations, thus reducing them to Eqs. (9) and (10), respectively, with $b = B_4/\sqrt{B_3}$.

We assume that Eq. (9) is elliptic. Then the inequality $b \geq 2$ must hold.

The axisymmetric solutions of Eq. (9) satisfy the equation

$$L_4 u \equiv u_{rrrr} + bu_{rrzz} + u_{zzzz} + \frac{2}{r}u_{rrr} + \frac{b}{r}u_{rzz} - \frac{1}{r^2}u_{rr} + \frac{1}{r^3}u_r = 0, \tag{11}$$

and the axisymmetric fundamental solutions [or the axisymmetric solutions of Eq. (10)] satisfy the equation

$$rL_4 u = \frac{1}{2\pi}\,\delta(r)\delta(z). \tag{12}$$

Here $r = \sqrt{x^2 + y^2}$ and $\int_0^\infty \delta(r)\,dr = 1$.

Equation (12) can also be represented in the divergence form

$$\left(ru_{rrr} + bru_{rzz} + u_{rr} - \frac{1}{r}u_r\right)_r + (ru_{zzz})_z = \frac{1}{2\pi}\,\delta(r)\delta(z). \tag{13}$$

SYMMETRIES OF THE MAIN EQUATIONS

In accordance with the preceding, we seek symmetry operators for Eq. (9) in the form

$$X = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial u},$$

where $\xi^1 = \xi^1(x, y, z)$, $\xi^2 = \xi^2(x, y, z)$, $\xi^3 = \xi^3(x, y, z)$, and $\eta = \eta(x, y, z, u)$. We use the invariance criterion in [3] to obtain the system of determining equations

$$\begin{aligned}
&\xi^1_u = 0, \qquad \xi^2_u = 0, \qquad \xi^3_u = 0, \qquad \eta_{uu} = 0,\\
&\xi^1_x - \xi^3_z = 0, \qquad \xi^1_y + \xi^2_x = 0, \qquad \xi^2_y - \xi^3_z = 0, \qquad 2\xi^1_z + b\xi^3_x = 0,\\
&b\xi^1_z + 2\xi^3_x = 0, \qquad 2\xi^2_z + b\xi^3_y = 0, \qquad b\xi^2_z + 2\xi^3_y = 0,\\
&6\xi^1_{xx} + 2\xi^1_{yy} + b\xi^1_{zz} - 4\eta_{xu} = 0, \qquad 2\xi^1_{xx} + 6\xi^1_{yy} + b\xi^1_{zz} + 8\xi^2_{xy} - 4\eta_{xu} = 0,\\
&2\xi^2_{xx} + 6\xi^2_{yy} + b\xi^2_{zz} - 4\eta_{yu} = 0, \qquad 6\xi^2_{xx} + 2\xi^2_{yy} + b\xi^2_{zz} + 8\xi^1_{xy} - 4\eta_{yu} = 0,\\
&b(\xi^1_{yz} + \xi^2_{xz}) + 2\xi^3_{xy} = 0, \qquad b(\xi^3_{xx} + \xi^3_{yy}) + 6\xi^3_{zz} - 4\eta_{zu} = 0,\\
&6\xi^3_{xx} + 2\xi^3_{yy} + b\xi^3_{zz} + 4b\xi^1_{xz} - 2b\eta_{zu} = 0, \qquad 2\xi^3_{xx} + 6\xi^3_{yy} + b\xi^3_{zz} + 4b\xi^2_{yz} - 2b\eta_{zu} = 0,\\
&\eta_{xxxx} + 2\eta_{xxyy} + \eta_{yyyy} + b(\eta_{xxzz} + \eta_{yyzz}) + \eta_{zzzz} = 0.
\end{aligned}$$

Now we analyze the solution of the system of determining equations and obtain the following assertion.

Proposition 2. *Equation* (9) *with an arbitrary parameter* b *admits the following basis of the Lie algebra of symmetry operators*:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial z}, \qquad X_4 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},$$
$$X_5 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \qquad X_6 = u\frac{\partial}{\partial u}, \qquad X_\infty = \varphi(x, y, z)\frac{\partial}{\partial u}.$$

For $b = 2$, *the basis of the Lie algebra is extended by the symmetry operators*

$$X_7 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \qquad X_8 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$
$$X_9 = (x^2 - y^2 - z^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} + 2xz\frac{\partial}{\partial z} + xu\frac{\partial}{\partial u},$$
$$X_{10} = 2xy\frac{\partial}{\partial x} + (y^2 - x^2 - z^2)\frac{\partial}{\partial y} + 2yz\frac{\partial}{\partial z} + yu\frac{\partial}{\partial u},$$
$$X_{11} = 2xz\frac{\partial}{\partial x} + 2yz\frac{\partial}{\partial y} + (z^2 - x^2 - y^2)\frac{\partial}{\partial z} + zu\frac{\partial}{\partial u}.$$

Here $u = \varphi(x, y, z)$ *is an arbitrary solution of Eq.* (9).

Let us find the symmetries of Eq. (10). Let us use the results in [4]. We use the finite-dimensional part of the Lie algebra of symmetry operators of Eq. (9) and consider a symmetry operator of the general form

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6.$$

Here the a_i $(i = 1, \ldots, 6)$ are arbitrary constants.

Proposition 3. $\underset{4}{X} L_3u = (a_6 - 4a_5)L_3u.$

Indeed, since $\lambda = a_6 - 4a_5$, it follows from Theorem 1 that $a_1 = a_2 = a_3 = 0$ and $a_5 - a_6 = 0$. Hence we obtain the following assertion.

Proposition 4. *Equation* (10) *with an arbitrary parameter* b *admits the following basis of the Lie algebra of symmetry operators*:

$$Y_1 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \qquad Y_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}. \tag{14}$$

Remark 2. One can also show that if $b = 2$, then Eq. (10) admits the symmetry operators (14) and the symmetry operators $X_7, \ldots, X_{11}$.

INVARIANT FUNDAMENTAL SOLUTION. MAIN RESULT

Let us find a solution of Eq. (9) invariant under the symmetry operators (14). The invariants of the admitted transformation group are $J_1 = r^2/z^2 = \tau$ and $J_2 = u/z$. Then we seek an invariant solution in the form

$$u = zf(\tau). \tag{15}$$

We substitute the expression (15) into Eq. (9) [or into Eq. (11)] and obtain the fourth-order ordinary differential equation

$$4\tau^2(\tau^2 + b\tau + 1)\frac{d^4f}{d\tau^4} + 2\tau(14\tau^2 + 11b\tau + 8)\frac{d^3f}{d\tau^3} + (39\tau^2 + 22b\tau + 8)\frac{d^2f}{d\tau^2} + 2(3\tau + b)\frac{df}{d\tau} = 0. \tag{16}$$

Proposition 5. *The ordinary differential equation* (16) *has the fundamental solution system*

$$\begin{aligned} f_1 &= 1, \\ f_2 &= \sqrt{\frac{\tau}{a}+1} - \operatorname{arctanh}\sqrt{\frac{\tau}{a}+1} + \sqrt{a\tau+1} - \operatorname{arctanh}\sqrt{a\tau+1}, \\ f_3 &= \frac{a}{a^2-1}\left(\sqrt{\frac{\tau}{a}+1} - \operatorname{arctanh}\sqrt{\frac{\tau}{a}+1} - \sqrt{a\tau+1} + \operatorname{arctanh}\sqrt{a\tau+1}\right), \\ f_4 &= \frac{a}{a^2-1}\biggl(\sqrt{\frac{\tau}{a}+1}\operatorname{arctanh}\sqrt{\frac{\tau}{a}+1} - \frac{1}{2}\operatorname{arctanh}^2\sqrt{\frac{\tau}{a}+1} \\ &\quad - \sqrt{a\tau+1}\operatorname{arctanh}\sqrt{a\tau+1} + \frac{1}{2}\operatorname{arctanh}^2\sqrt{a\tau+1}\biggr), \end{aligned}$$

where the parameter a satisfies the relation $b = a + 1/a$.

Consider the general solution

$$f = \sum_{i=1}^{4} c_i f_i \tag{17}$$

of Eq. (16), where the c_i ($i = 1, \ldots, 4$) are arbitrary constants. Of the solutions (17), we select solutions that, together with their first derivative, take a finite value at $\tau = 0$.

Proposition 6. *The following expansions hold as $\tau \to 0$:*

$$f = \left[c_2 + \frac{a \ln a}{2(a^2-1)}\, c_4\right] \ln\tau + O(1), \qquad \frac{df}{d\tau} = \left[c_2 + \frac{a \ln a}{2(a^2-1)}\, c_4\right] \frac{1}{\tau} + \frac{c_4}{8}\ln\tau + O(1).$$

Hence it follows that $c_2 = 0$ and $c_4 = 0$. In addition, set $c_1 = 0$. As a result, we obtain the following one-parameter family of solutions of Eq. (16) :

$$f = \frac{c_3 a}{a^2-1}\left(\sqrt{\frac{\tau}{a}+1} - \sqrt{a\tau+1} - \operatorname{arctanh}\sqrt{\frac{\tau}{a}+1} + \operatorname{arctanh}\sqrt{a\tau+1}\right).$$

Then we use the expression (15) and obtain the following one-parameter family of solutions of Eq. (9) :

$$u = \frac{c_3 a}{a^2-1}\left(\sqrt{\frac{r^2}{a}+z^2} - \sqrt{ar^2+z^2} - z\operatorname{arctanh}\frac{\sqrt{a^{-1}r^2+z^2}}{z} + z\operatorname{arctanh}\frac{\sqrt{ar^2+z^2}}{z}\right). \tag{18}$$

Let us show that the family (18) of solutions of Eq. (11) contains a fundamental solution. To this end, we integrate both sides of Eq. (13) over the rectangular domain

$$\Pi = \{(r,z) :\ 0 \le r \le r_0,\ \ -z_1 \le z \le z_2,\ r_0 > 0,\ z_1 > 0,\ z_2 > 0\}.$$

We use the Stokes formula to represent the integral of the left-hand side via an integral along the boundary of Π and substitute the solution (18) into the integrand. As a result, we obtain $c_3 = 1/(4\pi)$.

Let us state the main result of the present paper.

Theorem 2. *The fundamental solution of Eq.* (9) *has the form*

$$u_{\mathrm{f}} = \frac{a}{4\pi(a^2-1)}\left[\sqrt{\frac{r^2}{a}+z^2} - \sqrt{ar^2+z^2} + \frac{z}{2}\ln\frac{(\sqrt{a^{-1}r^2+z^2}-z)(\sqrt{ar^2+z^2}+z)}{(\sqrt{a^{-1}r^2+z^2}+z)(\sqrt{ar^2+z^2}-z)}\right]. \tag{19}$$

Remark 3. For $a = 1$ (or $b = 2$), the fundamental solution (19) acquires the form

$$u_{\mathrm{f}} = -\frac{1}{8\pi}\sqrt{r^2 + z^2}$$

and coincides with the fundamental solution of the biharmonic equation [7, p. 205].

Remark 4. For $b = 2$, the symmetry-based construction of the fundamental solution is especially efficient. In this case, the solution of Eq. (9) invariant under the symmetry operators (14) and the symmetry operators X_7 and X_8 is defined modulo a constant factor and has the form

$$u = c\sqrt{r^2 + z^2}. \tag{20}$$

[In this case, the transformation group admitted by Eq. (10) has the only invariant $J = u/\sqrt{r^2 + z^2}$.] By analogy with the preceding, we find that the solution (20) is a fundamental solution for $c = -1/(8\pi)$.

CONCLUSION

The construction of an invariant fundamental solution of the equation of a transversely isotropic linearly elastic medium in elementary functions is the main result of the present paper. Note that the symmetry-based approach to the construction of fundamental solutions can also be efficiently used for the construction of fundamental solutions of linear partial differential equations with variable coefficients and of higher-order equations.

ACKNOWLEDGMENTS

The research was supported by the Russian Foundation for Basic Research (projects nos. 15-01-00361 and 15-01-04677).

REFERENCES

1. Georgievskii, D.V., An extended Galerkin representation for a transversely isotropic linearly elastic medium, *J. Appl. Math. Mech.*, 2015, vol. 79, no. 6, pp. 618–621.
2. Georgievskii, D.V., The Galerkin tensor operator, reduction to tetraharmonic equations, and their fundamental solutions, *Dokl. Phys.*, 2015, vol. 60, no. 8, pp. 364–367.
3. Ovsyannikov, L.V., *Gruppovoi analiz differentsial'nykh uravnenii* (Group Analysis of Differential Equations), Moscow: Nauka, 1978.
4. Aksenov, A.V., Symmetries of linear partial differential equations, and fundamental solutions, *Dokl. Math.*, 1995, vol. 51, no. 3, pp. 329–331.
5. Aksenov, A.V., Symmetries and fundamental solutions of the multidimensional generalized axisymmetric Laplace equation, *Differ. Equations*, 1995, vol. 31, no. 10, pp. 1663–1667.
6. Bluman, G., Simplifying the form of Lie groups admitted by a given differential equation, *J. Math. Anal. Appl.*, 1990, vol. 145, no. 1, pp. 52–62.
7. Vladimirov, V.S., *Obobshchennye funktsii v matematicheskoi fizike* (Generalized Functions in Mathematical Physics), Moscow: Nauka, 1979.