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The surface tension effect on viscous liquid spreading along a superhydrophobic surface

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Abstract. Within the Stokes film approximation, unsteady plane-parallel spreading of a thin layer of a heavy viscous fluid along a horizontal superhydrophobic surface is studied. The forced spreading regimes induced by the mass supply are considered. Plane-parallel flow along the principal direction of the slip tensor of the superhydrophobic surface is studied in case that the corresponding slip tensor component is a power function of the spatial coordinate. An evolution equation for the film thickness is derived taking into account surface tension that is dependent on the spatial coordinate. The group classification problem is solved. Self-similar and invariant solutions are constructed for power and exponent time dependences on mass supply respectively at a special form of the surface tension coefficient. Surface tension is shown to have a significant influence on the character of the liquid spreading.

1. Introduction

For the analysis of mathematical models describing various physical processes the problem of finding exact solutions of the equations governing them is important [1]-[4]. The problem of Stokes film spreading from a localized source in the gravity field with the no-slip condition and its self-similar solutions were studied in many works. Steady flows along horizontal [5] and inclined [6] planes and also on conic surfaces [7] were considered. Unsteady Stokes film spreading was studied in the papers [8, 9]. In [10] within the Stokes film approximation unsteady spreading of a thin layer of a heavy viscous fluid along a horizontal superhydrophobic surface with the Navier slip condition [11, 12] was investigated in the presence of a given localized mass supply. Also, the evolution equation for the film thickness was derived, the forced spreading regimes were considered and the self-similar solutions analyzed in [10]. The surface tension effect was neglected in the works stated above.

The problem of a thin layer spreading of a heavy viscous fluid along a horizontal superhydrophobic surface has been studied in this work taking into account the surface tension. The surface tension was shown to have a significant effect on the film spreading character.

2. Statement of the problem

Similar to [10], it is possible to receive the film thickness equation for a heavy viscous fluid moving along a horizontal superhydrophobic surface on the assumption that surface tension is varying. It has the form

$$\frac{\partial h}{\partial t} - \frac{\partial}{\partial x} \left[\left(\frac{h^3}{3} + bh^2 \right) \left(\frac{\partial h}{\partial x} - \frac{\partial}{\partial x} \left(D \frac{\partial^2 h}{\partial x^2} \right) \right) \right] = 0.$$
(1)

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Equation (1) is written in terms of dimensionless variables. Here, h = h(x, t) is the film thickness, b = b(x) is the effective slip length in the direction of spreading, D = D(x) is the surface tension coefficient. Note that the no-slip condition corresponds to the case b = 0 and boundary frictionless condition corresponds to the case $b \to \infty$ [12]. As in [10] we will consider superhydrophobic surfaces with $b(x) = Bx^{\delta}$, B > 0, $\delta > 0$, where x is a dimensionless distance from the source.

The statement of the problem for finding the surface shape h(x,t) is completed by the formulation of an integral condition following from the mass supply law and the requirement of zero thickness of the film at the leading front of the wetted area

$$\int_{0}^{x_{f}} h(x,t)dx = Q(t), \qquad h(x_{f},t) = 0.$$
(2)

Here, Q(t) is a known volume of the fluid in the film, $x_f = x_f(t)$ is a dimensionless coordinate of the leading front.

3. Group classification

We seek symmetry operators of equation (1) as

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial h} \,,$$

where $\xi^1 = \xi^1(x, t, h)$, $\xi^2 = \xi^2(x, t, h)$, $\eta = \eta(x, t, h)$. Applying the criterion of invariance [2] we obtain the system of the determining equations. The analysis of the system solution shows that the kernel of the symmetry operators consists of the symmetry operator

$$X_1 = \frac{\partial}{\partial t} \,.$$

The kernel of the symmetry operators has an expansion at a special type of the function $D(x) = Kx^2$, K = const

$$X_2 = x\frac{\partial}{\partial x} + (2 - 3\delta)t\frac{\partial}{\partial t} + \delta h\frac{\partial}{\partial h}.$$

4. Types of invariant solutions

We will consider the symmetry operator as $X = aX_1 + X_2$, a = const. The invariants of a transformation group corresponding to the symmetry operator X have the following appearance in accordance with the parameters a and δ

$$\begin{split} I_1 &= x(t+t_0)^{-\frac{1}{2-3\delta}}, \qquad I_2 = h(t+t_0)^{-\frac{\delta}{2-3\delta}}, \qquad t_0 = \frac{a}{2-3\delta}, \qquad \text{if } \delta \neq 2/3\,;\\ I_1 &= xe^{-\frac{t}{a}}, \qquad I_2 = he^{-\frac{2t}{3a}}, \qquad \text{if } \delta = 2/3, \quad a \neq 0\,;\\ I_1 &= t, \qquad I_2 = hx^{-\frac{2}{3}}, \qquad \text{if } \delta = 2/3, \quad a = 0\,. \end{split}$$

The relevant invariant solutions have a form

$$h = (t+t_0)^{\alpha} F(\eta), \quad \eta = \frac{x}{C(t+t_0)^{\beta}}, \quad \alpha = \frac{\delta}{2-3\delta}, \quad \beta = \frac{1}{2-3\delta}, \quad \delta \neq 2/3;$$
(3)

$$h = e^{\alpha t} F(\eta), \quad \eta = \frac{x}{Ce^{\beta t}}, \quad \alpha = \frac{2}{3a}, \quad \beta = \frac{1}{a}, \quad \delta = 2/3, \quad a \neq 0;$$

$$(4)$$

$$h = x^{\frac{2}{3}}F(t), \quad \delta = 2/3, \quad a = 0.$$
 (5)

We do not consider invariant solution (5) further since it does not satisfy condition (2) at the leading edge.

5. Self-similar solution

We consider invariant solution (3) at $t_0 = 0$. It is in accord with the one-parameter group of non-homogeneous dilations. Invariant solution (3) is self-similar since the symmetry operator corresponding to him specify one-parameter group of non-homogeneous dilations [2].

Since $F(\eta_f) = 0 \eta_f$ is constant, hence the law of mass supply in the film is a power function of time and it can be calculated as follows

$$Q(t) = At^{\gamma}, \qquad A = C \int_{0}^{\eta_{f}} F(\eta) d\eta = \text{const} > 0, \qquad \gamma = \alpha + \beta.$$
(6)

The value of the constant C is chosen so that the self-similar coordinate of the leading edge η_f is equal to unity. After the substitution of the self-similar form of the solution in equation (1) we obtain the ordinary differential equation

$$\begin{split} (K\eta^2 F^3 + 3BK\eta^{\delta+2}C^{\delta}F^2)F'''' &+ (6\eta^{\delta+2}C^{\delta}BKFF' + 3K\eta^2F^2F' + \\ &+ 4K\eta F^3 + 3BK\delta\eta^{\delta+1}C^{\delta}F^2 + 12BK\eta^{\delta+1}C^{\delta}F^2)F''' + \\ &+ (6K\eta F^2 + 12BK\eta^{\delta+1}C^{\delta}F)F'F'' + (2KF^3 - F^3 + \\ &+ 6\eta^{\delta}C^{\delta}BKF^2 + 6BK\delta\eta^{\delta}C^{\delta}F^2 - 3B\eta^{\delta}C^{\delta}F^2)F'' - (3F^2 + \\ &+ 6B\eta^{\delta}C^{\delta}F)F'^2 - (3C^2\beta\eta - 3B\delta\eta^{\delta-1}C^{\delta}F^2)F' + 3C^2\delta\beta F = 0 \,. \end{split}$$

After the substitution of the expression for α and β from (3) in the last ratio in (6) we obtain

$$\alpha = \frac{2\gamma - 1}{5}, \qquad \beta = \frac{3\gamma + 1}{5}, \qquad \delta = \frac{2\gamma - 1}{3\gamma + 1}. \tag{8}$$

Note that the following inequations are valid due to (8) and $\delta > 0$

$$\gamma > \frac{1}{2}\,, \qquad 0 < \delta < \frac{2}{3}\,, \qquad \alpha > 0\,.$$

Ordinary differential equation (7) has the fourth order. There is an singularity at the point $\eta = 1$ as F(1) = 0 (at the leading edge) at this point. We expand the required solution $F(\eta)$ in the asymptotic series in the neighborhood of $\eta = 1$ and take the first term of the series. After the substitution of the asymptotic expansion of the solution in equation (7) we obtain

$$F = \sqrt{\frac{8\beta C^{2-\delta}}{3BK}} (1-\eta)^{3/2} + o\left((1-\eta)^{3/2}\right) \,. \tag{9}$$

The study of equation (7) solution behavior at $\eta \ll 1$ on the assumption that an order of magnitude $F(\eta)$ is larger than an order of magnitude η^{δ} shows self-similar solution does not exist when $0 < K \leq 1/2$.

Equation (7) was solved numerically using the fourth-order Runge-Kutta method. Asymptotics (9) has been used in the numerical calculations to shift the boundary condition from the singular point by a small distance to the point $1 - \eta_0$. The value of the constant C in the law of motion of the leading front was found by the shooting method with iterations, which continued until the integral mass conservation law was satisfied at a required accuracy.

Figure 1 a) shows the results of the numerical solution of equation (7) for A = 1, $\gamma = 1, 5$, B = 0, 5 and several values of the parameter K.



Figure 1. Function $F(\eta)$: B = 0.5, continuous lines correspond to K = 1, dashed lines to K = 5, dotted lines to K = 10, $\gamma = 1.5$, a) self-similar solution, b) invariant solution

From Figure 1 a) it follows that the curves of the film shape obtained with fixed value of the parameter γ lie closer to the horizontal axis η at the parameter K increasing. Calculations have shown the value of the constant C is increasing at the parameter K rising, thus for the same time the leading edge of the film passes a greater distance at a larger value of the parameter K than at smaller.

6. Invariant solution

In this section we consider invariant solution (4). As before the value of the constant C is chosen so that the self-similar coordinate of the leading edge $\eta = \eta_f$ is equal to unity. The variation of the fluid volume in the film is described by the formula

$$Q(t) = Ae^{\gamma t}, \qquad A = C \int_{0}^{1} F(\eta) d\eta, \qquad \gamma = \alpha + \beta.$$
(10)

After the substitution of expression (4) in equation (1) we obtain the following ordinary differential equation

$$\frac{1}{3}K\eta^{2}F^{3}F'''' + BC^{2/3}K\eta^{8/3}F^{2}F'''' + \left(F + 2BC^{2/3}\eta^{2/3}\right)K\eta^{2}FF'F''' +
+ \frac{4}{3}K\eta F^{3}F''' + \frac{14}{3}BC^{2/3}\eta^{5/3}KF^{2}F''' + 2K\eta F^{2}F'F'' +
+ 4BC^{2/3}K\eta^{5/3}FF'F'' + \frac{2}{3}KF^{3}F'' - \frac{1}{3}F^{3}F'' +
+ \left(\frac{10}{3}K - 1\right)BC^{2/3}\eta^{2/3}F^{2}F'' - 2BC^{2/3}\eta^{2/3}FF'^{2} -
- F^{2}F'^{2} - \frac{3}{5}\gamma C^{2}\eta F' - \frac{2}{3}BC^{2/3}\eta^{-1/3}F^{2}F' + \frac{2}{5}\gamma C^{2}F = 0.$$
(11)

Analogously to the case of the self-similar solution formula (10) gives

$$\alpha = \frac{2\gamma}{5}, \qquad \beta = \frac{3\gamma}{5}$$

Ordinary differential equation (11) is a forth-order differential equation. The point $\eta = 1$ is singular as F(1) = 0 (at the leading edge). In the neighborhood of $\eta = 1$ the approximate solution has the form

$$F = \sqrt{\frac{8\gamma C^{4/3}}{5BK}} (1-\eta)^{3/2} + o\left((1-\eta)^{3/2}\right) \,.$$

The analysis of equation (11) solution behavior at $\eta \ll 1$ on the assumption that an order of magnitude $F(\eta)$ is larger than an order of magnitude η^{δ} shows invariant solution does not exist when $0 < K \leq 1/2$.

Equation (7) was solved numerically analogously to the procedure described in the previous section. Figure 1 b) shows the results of numerical solution of equation (11) for A = 1. For the same time the leading edge of the film passes a greater distance at a larger value of the parameter K than at smaller as in the case of the power law of mass supply.

7. Comparison of solutions taking into account surface tension and without it

Figure 2 shows the plots of self-similar solutions of the equation (1) taking into account surface tension and without it (for more detail, see [10]) at the given parameters of the problem.



Figure 2. B = 0.5, $\gamma = 2.0$, continuous lines correspond to K = 10, dashed lines to K = 0

Structures of the related flows are differ significantly. It is caused by the fact that the surface tension influences on the film spreading.

For the same time the leading front of a film passes various distances in cases when capillary effects are considered and not. It is necessary to know values of a constant C to compare these distances. In case of the account of the surface tension the constant C = 1.146 at K = 10 and C = 0.862 in case of the surface tension absence. Hence for the case presented in Figure 2 the leading front of the film spreading along a superhydrophobic surface taking into account the surface tension moves quicker than the leading front of the film spreading along a superhydrophobic surface without surface tension.

8. Conclusion

The equation for a film thickness of a heavy viscous liquid, which spreading is plane-parallel, is derived in the present work. The symmetries of the investigated equation are found. The surface tension coefficient form of the dependence on the coordinate is stated. There is the maximum Lie algebra expansion of the symmetry operators under the specified surface tension coefficient. Invariant solutions are found and investigated. the accounting of surface tension at the special selection of the surface tension coefficient is shown to cause a change of the current mode as opposed to a case with no surface tension. The surface tension coefficient has been chosen assuming that self-similarity of the problem is reserved. It allowed to carry out comparison with already known self-similar solution for the surface tension absence case. It is necessary to solve a nonisothermal problem for detailed research of the surface tension influence on a spreading mode of a viscous liquid. It will allow to determine the surface tension coefficient when it depends on temperature.

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