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Orientation Instability of Shear Flow of a Nematic Liquid Crystal

A. G. Kalugin
Moscow State University, Faculty of Mechanics and Mathematics, Leninskie Gory, Moscow, 119899, Russia; e-mail: kalugin@mech.math.msu.su

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Abstract—The instability of shear flow of a nematic liquid crystal layer is studied. The case when the orientation vector and the flow velocity vector are parallel is considered. It is shown that the orientation instability of this flow is possible if the anchoring boundary condition is weak and if the splay-bend constants in the Frank energy are taken into account. For this type of instability, periodic structures are possible to appear. Their wave vector belongs to the plane of flow and is perpendicular to the velocity vector. The medium parameters are estimated on the basis of the existence condition for this instability. The period of the appearing periodic structures is evaluated.

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An analog of Couette flow was obtained in [1] for the nematic liquid crystals when deriving the equations describing the corresponding model. Later on, the stability of such a flow was studied theoretically and experimentally. In [2–5] it is shown that, in the shear flows of nematic liquid crystals, various forms of instability may appear when the unit orientation vector $\mathbf{n}$ (also called the director) is given at the boundary. The case of weak anchoring is studied in [6, 7] when the orientation is specified at the boundary by the energy minimum condition. In [8] it is shown that the instability of Poiseuille flow may appear irrespective of boundary conditions imposed on the director. The effect of splay-bend constants on the stability of surface waves and on the stability of quiescent nematic layers is analyzed in [9–11].

In [12] it is shown that the instability of shear flows is possible when the velocity vector is not parallel to the director and when this director belongs to the plane of flow. The instability of the orientation vector is studied in [13, 14] when this vector is perpendicular to the walls at the boundary. In this paper we propose a model of instability evolution when the director is parallel to the velocity vector.

Below we consider the model of a nematic liquid crystal proposed in [1, 15]. The equations of motion and the equations of director evolution take the following form for an incompressible nematic when the mass forces are absent:

$$\nabla_i v^i = 0, \qquad \rho \frac{dv^i}{dt} + \nabla^i p + \nabla_j \left(\frac{\partial F_V}{\partial \nabla_j n^k} \nabla_i n^k \right) = \nabla_j \tau^{ij}, \tag{1}$$

$$\left(\delta_k^j - n^j n_k\right) \left(\gamma_1 N^i + \gamma_2 e^{ik} n_k - \nabla_i \left(\frac{\partial F_V}{\partial \nabla_i n^j} \right) \right) = 0\,, \qquad \gamma_1 = \alpha_3 - \alpha_2\,, \qquad \gamma_2 = \alpha_6 - \alpha_5, \tag{2}$$

$$\tau_{ij} = \alpha_2 n_i N_j + \alpha_3 n_j N_i + \alpha_4 e_{ij} + \alpha_5 e_{ik} n^k n_j + \alpha_6 e_{jk} n_i n^k\,, \qquad N^i = \frac{dn^i}{dt} - [\boldsymbol{\omega}, \mathbf{n}]^i\,, \qquad 2\boldsymbol{\omega} = \operatorname{rot} \mathbf{v}.$$

Here v^i are the velocity vector components, ρ is a constant density, p is pressure, F_V is the Frank free energy of director field distortions, τ^{ij} are the components of the viscous stress tensor, e_{ij} are the components of the strain-rate tensor, α_i and γ_i are the Leslie viscosity coefficients, δ_j^i is the Kronecker symbol, N^i is the Jaumann derivative expressing the variation of the director relative to the moving particle of the medium, $\boldsymbol{\omega}$ is the vortex vector, and ∇_i is the covariant derivative.

The equation expressed by (2) is projected onto the plane perpendicular to the director to eliminate the undetermined Lagrange multiplier arising because of the constant length condition [16]. The Frank energy is considered in the one-constant approximation [9, 15], which is taken into account in (2):

$$2F_V = K \nabla_i n_j \nabla^i n^j + K_{24} \left(\nabla_i n_j \nabla^j n^i - (\nabla_k n^k)^2 \right). \tag{3}$$

Here K and K_{24} are the constant Frank coefficients. The second term of (3) is of divergent form and, hence, does not influence the equations inside the volume; however, this term is used in the boundary conditions.

Let us consider a nematic liquid crystal layer bounded by the walls $z = \pm h$ given in the Cartesian coordinate system (x, y, z). If the upper wall moves with a constant velocity $\mathbf{v} = (V, 0, 0)$ and the lower wall is fixed, then there exists the stationary solution $\mathbf{v} = (k(z+h), 0, 0)$, $2hk = V$, $(n^3/n^1)^2 = \alpha_3/\alpha_2$, $n^2 = 0$ (see [1, 15]). Since the inequality $|\alpha_2| \gg |\alpha_3|$ is valid for many liquid crystals [15], we put $\alpha_3 = 0$. In the unperturbed state, hence, we have $\mathbf{n} = (1, 0, 0)$. The above solution satisfies the kinematic boundary conditions $v^i = 0$ and the following weak anchoring boundary condition for the director [6, 16]:

$$\left(\delta^j_k - n^j n_k\right)\left(\frac{\partial F_V}{\partial \nabla_i n^j} m_i + \frac{dF_S}{dn_m} m_j\right) = 0. \tag{4}$$

Here $2F_S = 2\gamma + W(\mathbf{n}, \mathbf{m})^{\mathbf{2}}$ is the Rapini–Papoular anchoring energy, γ and W are constant coefficients, and $\mathbf{m}$ is the unit normal to the surface. The minimum of the surface energy is achieved when the director is oriented along the walls.

Now we determine the perturbed solutions for the velocity and director. These solutions are dependent on y and z. The linearized equations (1) and (2) take the following form for the velocity perturbations u^i and for the pressure p^*:

$$\rho u^1_t = \frac{\alpha_4 + \alpha_6}{2}\,\Delta u^1\,, \quad \rho u^2_t + p^*_y = \frac{\alpha_4}{2}\,\Delta u^2\,, \quad \rho u^3_t + p^*_z = \frac{\alpha_4}{2}\,\Delta u^3\,, \quad u^2_y + u^3_z = 0, \tag{5}$$

$$-\alpha_2 n^2_t = \alpha_2 u^1_y + K\Delta n^2\,, \quad -\alpha_2 n^3_t = \alpha_2(k + u^1_z) + K\Delta n^3. \tag{6}$$

Here the subscripts t, y, and z indicate the partial derivatives with respect to the corresponding variables. The director perturbation along the x-axis is not considered, since the length of the director is constant.

The linearized boundary conditions (4) can be reduced to the following equations for $z = \pm h$:

$$Kn^2_z + K_{24} n^3_y = 0\,, \tag{7}$$

$$Kn^3_z = K_{24} n^2_y \mp Wn^3. \tag{8}$$

Here the condition expressed by (7) is valid on both the boundaries. In the condition expressed by (8), the minus sign corresponds to the lower boundary, whereas the plus sign corresponds to the upper boundary.

Usually, the following inequalities are valid: $\alpha_4 > |\alpha_6| > 0$ and $\alpha_2 < 0$ [15]. Hence, the solutions to Eqs. (5) decrease with time. In the case of 4-methoxybenzylidene-4'-butylaniline, the layer thickness h is equal to 10^{-4} m and the characteristic decrease time t_1 is $\rho h^2/\alpha_4 < 10^{-3}$ s. Hence, system (6) can be considered irrespective of (5) and the terms containing the derivatives of u^1 can be omitted. For (6) we seek nontrivial solutions periodic in y. Without loss of generality, we represent the solution to (6) as [17]

$$n^2 = (C_1 \exp(lz) + C_2 \exp(-lz)) \cos ly,$$

$$n^3 = -\alpha_2 k z^2/2 + D_1 z + D_2 + (C_3 \exp(lz) + C_4 \exp(-lz)) \sin ly,$$

where C_i and D_i are arbitrary constants, l is the wave number, and $l > 0$. From (7) and (8), we have $D_1 = 0$ and $D_2 = \alpha_2 lh(Wh + 2K)/(2W)$. The existence of nonzero values of C_i is ensured if the corresponding system of linear homogeneous equations is singular, i.e., if the following relations are valid:

$$l(K^2_{24} - K^2)\,\mathrm{sh}(lh) = KW\,\mathrm{ch}(lh)\,, \quad l(K^2_{24} - K^2)\,\mathrm{ch}(lh) = KW\,\mathrm{sh}(lh).$$

From these relations it follows that the perturbation period is equal to $2\pi/l$. In the first case, this period exists for $|K_{24}| > K$ and for an arbitrary thickness of the layer. In the second case, it is additionally required that the layer thickness should be greater than $h_c = (K^2_{24} - K^2)/(WK)$. Hence, the perturbations of the director increase with time and tend to a nontrivial periodic solution with the above critical wave numbers. As a result, the flow becomes orientationally unstable. Some periodic structures may appear in parallel to the velocity vector; such structures were observed in the experiments discussed in [13, 14]. For Eqs. (6) the characteristic time $t_2 = |\alpha_2|h^2/K$ may reach the values of order 10^3 s. This means that the instability evolution may take a lot of time.

In this paper, thus, we show that, for a shear flow in a nematic liquid crystal layer, the orientation instability is possible when the flow velocity vector is parallel to the director. We also find the parameters of the medium when this instability can be observed. A dependence of the period of appearing periodic structures on the layer thickness and on the Frank constants is found.

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