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Acceleration of Weak Shock Waves

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Abstract—Exact solutions can demonstrate possible ways of the behavior of gas dynamics solutions when the density of the medium, being initially in equilibrium, decreases. In the present study a method of solving the wave equation with variable speed of sound is designed within the framework of the acoustic approximation using an expansion in series in terms of the characteristic variable. It is shown that in each step there exists an integral of the equations of motion which makes it possible to express the solution in finite form. The motion is initiated by the impact of a piston which generates a weak accelerated shock wave. The presence of a homogeneous gravity field is taken into account.

Keywords: shock wave, acceleration, gravity field, asymptotic behavior.

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The processes of acceleration of shock waves due to decrease in the initial density can occur in both star's and planetary atmospheres subjected to local heating or ionization. Already Sedov [1] revealed this effect within the framework of gas dynamics when solving the problem of point explosion in a medium with variable density when there is no counterpressure. On the other hand, taking the initial constant pressure into account, a decrease in the density automatically leads to an increase in the sonic speed and, consequently, in the shock wave velocity, i.e., prerequisites for the loss of inertiality of the medium, the instability, and the development of various dynamic processes are produced.

A very simple example of an exact solution can be presented; it concerns with the problem of a piston which starts to move at a constant velocity in a pressureless gas and produces an accelerated shock wave when a certain law of decrease in the initial equilibrium density is fulfilled [2, problem 25.37]. In a more realistic situation the effects of counterpressure and electromagnetic and gravity fields should be taken into account, as well as relativity theory. The exact solutions of this problem within the framework of the special and general relativity theory but without regard for the counterpressure were given in [3], while the results with taking the counterpressure into account within framework of the special theory but without gravity were announced in [4] and those with account for the frozen-in transverse magnetic field and the counterpressure in the Newtonian mechanics were given in [5, 6] and in a gravity field in [7]. In [8] the complete investigation of the class of self-similar problems with homogeneous gravity field when the initial density decreases in accordance with the power-law was carried out. In [9] a general review of earlier studies can be found.

1. EQUATIONS AND RELATIONS ON A DISCONTINUITY

We will consider the class of solutions of one-dimensional adiabatic flow of a perfect gas with plane waves in a homogeneous gravity field g within the frameworks of Newtonian mechanics. We will use the Lagrangian mass coordinate m.

Let $x(m,\ t)$ be the law of motion, $v = x_t$ be the velocity, $\rho = 1/x_m$ be the density, $p = f(m)\rho^\gamma$ be the pressure, and γ be a constant specific heat ratio. The subscripts t and m denote the partial derivatives.

In these variables the complete equation of motion takes the form:

$$v_t + p_m + g = 0. \tag{1.1}$$

Here, the x axis is directed counter the action of the gravity force.

The relations on discontinuity $m = m_s(t)$ give

$$\begin{gathered}[x]_0^1 = 0, \qquad [v\dot{m}_s - p]_0^1 = 0, \\ \left[\left(\frac{v^2}{2} + \frac{p}{(\gamma - 1)\rho}\right)\dot{m}_s - pv\right]_0^1 = 0,\end{gathered} \tag{1.2}$$

where zero denotes the initial equilibrium state and unity denotes the state downstream of the discontinuity.

By virtue of the equilibrium equation $p_0 = g(m_0 - m)$, where m_0 is the total gas layer mass (calculated per unit cross-sectional area). The function $x_0(m)$ is arbitrary.

Linearization of the equation of motion (1.1) about the equilibrium state with respect to the variable $u(m, t) = x - x_0$ and use of the assumption of continuity of the entropy function $f(m)$ (this is fulfilled correct to $(p_1 - p_0)^2$ inclusively [10]) give the equation

$$u_{tt} - (b_0^2(m)u_m)_m = 0, \tag{1.3}$$

where $b_0^2 = \gamma p_0 \rho_0$ is the square of the undisturbed speed of sound with respect to the mass which is continuous. The initial density is $\rho_0 = 1/x_m^0$. The usual speed of sound is $a_0 = b_0/\rho_0$.

On the discontinuity we have

$$u = 0, \qquad u_t\dot{m}_s - b_0^2 u_m = 0. \tag{1.4}$$

Here, the wave propagating to the right is considered and the quantity $\dot{m}_s$ is positive. With allowance for $u = 0$ the second relation shows that the velocity of a weak shock wave with respect to the mass is equal to $\dot{m}_s = b_0$.

In addition, it is necessary to specify the law of the piston motion when $m = 0$, namely, $u(0, t) = u_p(t)$. The latter function will be assumed to be analytic. Thus, we have the typical characteristic-boundary-value problem for Eq. (1.3), which could be solved for a series of special functions $b_0(m)$ by constructing the Riemann function. However, in what follows we will use the method of expansion in the Taylor series in terms of the characteristic variable which makes it possible explicitly to solve the ordinary differential equations for the coefficients and investigate convergence for large x.

2. METHOD OF SOLVING

It is convenient to go over to the variables

$$\tau = t - \mu, \qquad \mu = t_s(m) = \int_0^m \frac{dm}{b_0}.$$

The function $t_s(m)$ is inverse to the function $m_s(t)$. Let $\tau = 0$ be a characteristic along which the shock wave moves. Then on this characteristic we have

$$u = 0, \qquad u_\mu = 0, \quad u_{\mu\mu}, \ldots \tag{2.1}$$

It is clear that $u_\mu = 0$ is the approximate momentum conservation law (1.4). Similarly, the energy conservation law (the last of relations (1.2)) must also be satisfied.

When $\mu = 0$ the condition on the piston is $u = u_p(\tau)$.

In these variables we have

$$(\sqrt{b_0}\, u_\tau)_\mu = \frac{1}{2\sqrt{b_0}} \big(b_0 u_\mu\big)_\mu. \tag{2.2}$$

When $\tau = 0$ with allowance for (2.1) from Eq. (2.2) there follows an integral on the characteristic

$$v_1 = u(0,\ \mu)_\tau = \frac{C_1}{\sqrt{b_0}}. \tag{2.3}$$

This determines jump in velocity on the shock wave for any $b_0(m)$. As $b_0 \to 0$, the velocity $v_1 \to \infty$. Using (2.3), we will seek the solution in the form of a series

$$u = \sum_{n=1}^{\infty} \frac{\tau^n}{n!} u_\tau^{(n)}(0,\ \mu).$$

Differentiating (2.2) $n - 1$ times with respect to τ, we obtain

$$(\sqrt{b_0} u_\tau^{(n)})_\mu = \frac{1}{2\sqrt{b_0}} \big(b_0 u_{\tau,\mu}^{(n-1)}\big)_\mu.$$

In particular, jump in the acceleration is equal to

$$a_1 = u_{\tau\tau} = \frac{1}{\sqrt{b_0}} \left(C_2 + \int\limits_0^\mu \frac{1}{2\sqrt{b_0}} (b_0 v')'\, d\mu \right).$$

For power functions $\rho_0 \sim x^{-\omega}$

$$a_1 = \frac{1}{\sqrt{b_0}} \left(C_2 + \frac{C_3}{\sqrt{x}} \right)$$

etc.

For large x

$$v \approx \dot{u}_p(\tau) \frac{\sqrt{b_0(0)}}{\sqrt{b_0(\mu)}}.$$

Thus, the boundary condition on the piston are restored as $x \to \infty$ with multiplying by the known function μ.

We note that the next approximation is not considered here. In this approximation a small addition to the shock wave velocity $D = a_0 + v_1\,(\gamma+1)/4$ is taken into account. This addition leads to decay in jump in the velocity: $v_1 \sim 1/\sqrt{t}$ [1, 10].

This correction is actually based on the approximation of the exact solution of the gas dynamics equations with a weak discontinuity: $v = 2(x/t - a_0)/(\gamma + 1)$ for constant a_0 regardless for gravitation. The problem is sharply complicated on the variable background and, obviously, so far it has not been studied.

The applicability of the linear approximation is determined by the inequality $a_0 \gg v$ or, taking the equilibrium equation into account, by the inequality

$$p_0^3 \gg \mathrm{const}\, \rho_0.$$

In this case the inequalities $1 < \omega \le 3/2$ must be satisfied for $\rho_0 \sim x^{-\omega}$ $x \to \infty$. The nonlinear theory must be used for the greater ω. Moreover, this is necessary for exponential distributions of the type $\rho_0 \sim \exp(-kx)$. This is usually accepted in literature concerning the atmosphere acoustics by virtue of $b_0'/b_0 =$ const.

However, there are exact nonlinear solutions (see, e.g., [5, 6]) in which asymptotics (2.3) are fulfilled. In particular, for $g = 0$, constant p_0, and the solution with separation of variables $x = v(\xi)t$, where $\xi = x_0$, as $\xi \to \infty$, the following relations hold

$$\rho_0 \sim \xi^{-4}, \qquad a_0 \sim \xi^2, \qquad v \sim \xi$$

when the shock wave goes to infinity in finite time.

We note that for any arbitrary $b_0(\mu)$ equation (2.2) has solutions in the form of a polynomial of a given degree in terms of τ.

3. A NONLINEAR PROBLEM

We can give another very simple example of the solution of nonlinear equations (1.1) with the same asymptotics at infinity. Let a piston move with a constant acceleration A initiating the solid-body gas flow

$$x = \frac{At^2}{2} + w(m), \qquad p = \frac{f(m)}{w_m^\gamma} = (A + g)(m_0 - m).$$

The momentum and energy conservation conditions (1.2) give $t_s = C/(m_0 - m)$, where $C = \text{const}$ and the shock wave velocity

$$D = \left(\frac{\gamma + 1}{2}A + \gamma g\right)t.$$

We note that, as $A \to 0$, the velocity $D \to \gamma g t$.

From continuity of the law of motion there follows

$$x_0 = \left(\frac{\gamma + 1}{2}A + \gamma g\right)\frac{C^2}{2(m_0 - m)^2}.$$

In this case the initial density $\rho_0 = 1/x_m^0 \sim x^{-3/2}$. Thus, the same asymptotic law (2.3) $v_1^2 a_0 \rho_0 \sim 1$ must be satisfied on the shock wave.

4. THREE-DIMENSIONAL PROBLEMS

The above method can be generalized to include three-dimensional problems.

The vector equations of adiabatic flow of a perfect gas written in the Lagrangian form in the presence of a homogeneous gravity field $\mathbf{g} = (0,\ 0,\ -g)$ have the form:

$$\begin{gathered} \rho_0(\xi)x_{i,tt} + |x_\xi|\xi_i^\alpha p_\alpha - \rho_0(\xi)g_i = 0, \\ p|x_\xi|^\gamma = f(\xi), \qquad (|x_\xi|\xi_i^\alpha)_\alpha \equiv 0, \end{gathered} \tag{4.1}$$

where $x^i = x^i(\xi^\alpha,\ t)$ the law of motion in a Cartesian coordinate system, $x^i(\xi,\ 0) = \xi^i$ are the Lagrangian coordinates, $|x_\xi|$ is the determinant of the matrix $(\partial x^i/\partial \xi^\alpha)$, and (ξ_i^α) is the inverse matrix, $i,\ \alpha = 1,\ 2,\ 3$.

An equilibrium background gives

$$p_0'(\xi^3) = -\rho_0(\xi^3)g.$$

Linearization of (4.1)

$$x^i = \xi^i + u^i(\xi, t), \qquad p = p_0 + q(\xi, t), \quad f(\xi) = f_0$$

leads to the equations

$$\begin{gathered} q = -\gamma p_0 \nabla_i u^i, \qquad a_0^2 = \gamma p_0/\rho_0, \\ u_{i,tt} - a_0^2 \nabla_k \nabla_i u^k + g(\delta_i^3(\gamma - 1)\nabla_k u^k + \nabla_i u^3) = 0. \end{gathered} \tag{4.2}$$

The characteristics $\tau = t - \varphi(\xi) = \text{const}$ are determined by the general solution of the following Hamiltonian system (family of bicharacteristics) with the Hamiltonian $H = 0$

$$2H = |\mathbf{p}|^2 - \frac{1}{a_0^2(\xi^3)}, \qquad p_i = \nabla_i \varphi,$$

which has the form:

$$\frac{d\xi^i}{ds} = p^i, \qquad \frac{dp_\sigma}{ds} = 0, \qquad \sigma = 1,\ 2,$$

$$\frac{dp_3}{ds} = -\frac{a_0'(\xi^3)}{a_0^3},$$

and the relation

$$\frac{d\varphi}{ds} = p_i \frac{d\xi^i}{ds}$$

after elimination of the parameter s and two arbitrary constants.

As a result, we obtain

$$\varphi = \int\limits_0^{\xi^3} \frac{d\zeta}{a_0\big(1 - a_0^2(C_1^2 + C_2^2)\big)^{1/2}} + C_0(C_\sigma) + C_\sigma \xi^\sigma,$$

$$\frac{\partial \varphi}{\partial C_\sigma} = 0.$$

In addition, we can determine the Lagrangian coordinates of the family of bicharacteristics $\psi^\sigma = \xi^\sigma - C^\sigma s$:

$$s = \int\limits_0^{\xi^3} \frac{a_0 d\zeta}{\big(1 - a_0^2(C_1^2 + C_2^2)\big)^{1/2}}.$$

For the power-functions $\rho_0 \sim (\xi^3)^{-\omega}$, when the speed of sound is proportional to $a_0 \sim \sqrt{\xi^3}$, the integrals can be evaluated. In this case we can even assume the heat-conduction law is fulfilled for the initial temperature T_0 which is proportional to $a_0^2 = \gamma R T_0$.

In what follows, we will use the coordinates τ, ξ^α, the normal to the characteristic is $\mathbf{n} = a_0 \nabla \varphi$, and the tangential velocity is $\mathbf{v}_{tan}$. Dot denotes the derivative with respect to τ. Then equations (4.2) take the form:

$$\begin{gathered} \dot{v}^i_{tan} + a_0^2(\nabla_i(\nabla_k \varphi \dot{u}^k) + \nabla_i \varphi \nabla_k \dot{u}^k - \nabla_i \nabla_k u^k) \\ + g_i(\gamma - 1))\nabla_k \varphi \dot{u}^k + g_k \nabla_i \varphi \dot{u}^k - g_i(\gamma - 1)\nabla_k u^k - g_k \nabla_i u^k = 0. \end{gathered} \tag{4.3}$$

On the characteristic $\tau = 0$ the displacement vector is $u^i = 0$ and also, by virtue of $\mathbf{v}_{tan} = 0$ the total velocity vector is $\dot{u}^i = n^i v_n$.

If we use the derivative $d/ds = \nabla^i \varphi \nabla_i$, then equation (4.3) projected to the normal gives the linear equation

$$\frac{dv_n}{ds} + \frac{1}{2}\big(\Delta\varphi - \gamma g \nabla_3 \varphi |\nabla \varphi|^2\big) v_n = 0.$$

Its integral has the form:

$$v_n = Q(\psi^\sigma) \exp\left(-\frac{1}{2} \int\limits_0^s (\Delta\varphi - \gamma g \nabla_3 \varphi |\nabla\varphi|^2)\, ds \right). \tag{4.4}$$

This relation is a generalization of the integral (2.3).

The projection of Eq. (4.3) on the direction tangential to the discontinuity leads to the following equation for the tangential velocity

$$\dot{v}^i_{tan} + \left(a_0^2 \nabla_j \frac{v_n}{a_0} + g_j(\gamma - 1)\frac{v_n}{a_0}\right)(\delta^{ij} - n^i n^j) = 0,$$

which can readily be integrated using the integral (4.4).

A weak perturbation of the characteristics of flow with plane waves makes it possible to simplify the expression for the function $\varphi = \varphi_0(\xi^3) + \chi(\psi^\sigma)$ and investigate the arising features of the solution in more detail.

It is clear that for increasing speed of sound ahead of the shock wave accelerated as a whole even a weak distortion of its initial shape, for example, sinusoidal, will lead to the advanced growth of “tongues” and the formation of trailing dips. This can also be accompanied by formation of caustics related to appearance of enveloping curves and subsequent intersection of characteristics.

The solution can similarly be constructed for any static field $\mathbf{g}(x^i)$ which must satisfied the equilibrium conditions $\mathbf{g} \cdot \mathrm{curl}\, \mathbf{g} = 0$. In this case it is necessary to take into account the term $\rho_0 u^k \nabla_k g_i$ in the disturbed equations of motion (4.2).

Summary. The presence of universal asymptotics of the behavior of accelerated shock waves, namely, production of the square of jump in the velocity, the initial speed of sound, and the density remain constant is demonstrated referring for both simple examples of exact solutions of adiabatic flow of a perfect gas with decreasing initial density and also within the frameworks of the acoustic approximation for one- and three-dimensional flows.

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