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MOTION OF FLUIDS WITH VERY LITTLE VISCOSITY

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MOTION OF FLUIDS WITH VERY LITTLE VISCOSITY.*

By L. Prandtl.

In classic hydrodynamics the motion of nonviscous fluids is chiefly discussed. For the motion of viscous fluids, we have the differential equation whose evaluation has been well confirmed by physical observations. As for solutions of this differential equation, we have, aside from unidimensional problems like those given by Lord Rayleigh (Proceedings of the London Mathematical Society, 11 page 57 = Papers I page 474 ff.), only the ones in which the inertia of the fluid is disregarded or plays no important role. The bidimensional and tridimensional problems, taking viscosity and inertia into account, still await solution. This is probably due to the troublesome properties of the differential equation. In the "Vector Symbolics" of Gibbs,** this reads

$$\rho \left(\frac{\partial v}{\partial t} + v \circ \Delta v \right) + \Delta (V + p) = k \Delta^2 v \qquad (1)$$

in which v is the velocity; ρ, the density; V, a function of the power; p, pressure; k, viscosity constant. There is also the continuity equation

$$\text{div } v = 0.$$

for incompressible fluids, which alone will be here considered.

From the differential equation, it is easy to infer that, for sufficiently slow and also slowly changing motions, the factor ρ, in contrast with the other terms, can be as small as desired, so that the effect of the inertia can here be disregarded with sufficient approximation. Conversely, with sufficiently rapid motion, the quadratic term $v \, \rho \, \Delta v$ (change of velocity due to change of location) is large enough to let the viscosity effect appear quite subordinate. The latter almost always happens in cases of fluid motion occurring in technology. It is therefore logical simply to use here the equation for non-viscous fluids. It is known, however, that the solutions of this equation generally agree very poorly with experience. I will recall only the Diréchlet sphere, which, according to the theory, should move without friction.

I have now set myself the task to investigate systematically the laws of motion of a fluid whose viscosity is assumed to be very small. The viscosity is supposed to be so small that it can be disregarded wherever there are no great velocity differences nor accumulative effects. This plan has proved to be very fruitful, in that, on the one hand, it produces mathematical formulas, which enable a solution of the problems and, on the other hand, the agreement with observations promises to be very satisfactory. To mention one instance now: when, for example, in the steady motion around a sphere, there is a transition from the motion with viscosity to the limit of nonviscosity, then something quite dif-

ferent from the Dirichlet motion is produced. The latter is then only an initial condition, which is soon disturbed by the effect of an ever-so-small viscosity.

I will now take up the individual problems. The force on the unit area, due to the viscosity, is

$$K = k \, \Delta^2 \, v \qquad\qquad (2)$$

If the vortex is represented by $w = \frac{1}{2}$ rot v, then $K = -2 \, k$ rot w, according to a well-known vector analytical transformation, taking into consideration that div v = 0. From this it follows directly that, for w = 0, also K = 0, that is, that however great the viscosity, a vortexless flow is possible. If, however, this is not obtained in certain cases, it is due to the fact that turbulent fluid from the boundary is injected into the vortexless flow.

With a periodic or cyclic motion, the effect of viscosity, even when it is very small, can accumulate with time. For permanence, therefore, the work of K, that is, the line integral $\int K \circ d s$ along every streamline with cyclic motions, must be zero for a full cycle.

$$\int_P K \circ d s = (V_2 + p_2) - (V_1 + p_1).$$

A general formula for the distribution of the vortex can be derived from this with the aid of the Helmholtz vortex laws for bidimensional motions which have a flow function ψ (Cf. "Encyklopadie der mathematischen Wissenschaften," Vol. IV, 14, 7).

With steady flow we obtain*

$$- \frac{dw}{d\psi} = \frac{(V_2 + p_2) - (V_1 + p_1)}{2k \int v \circ d s}$$

With closed streamlines this becomes zero. Hence we obtain the simple result that, within a region of closed streamlines, the vortex assumes a constant value. For axially symmetrical motions with the flow in meridian planes, the vortex for closed stream-lines is proportional to the radius $w = cr$. This gives a force $K = 4 kc$ in the direction of the axis.

The most important aspect of the problem is the behavior of the fluid on the surface of the solid body. Sufficient account can be taken of the physical phenomena in the boundary layer between the fluid and the solid body by assuming that the fluid adheres to the surface and that, therefore, the velocity is either zero or equal to the velocity of the body. If, however, the viscosity is very slight and the path of the flow along the surface is not too long, then the velocity will have its normal value in immediate proximity to the surface. In the thin transition layer, the great velocity differences will then produce noticeable effects in spite of the small viscosity constants.

This problem can be handled best by systematic omissions in the general differential equation. If k is taken as small in

*According to Helmholtz, the vortex of a particle is permanently proportional to its length in the direction of the vortex axis. Hence we have, with steady even flow on each streamline ($\psi = $ const.), w constant, consequently $w = f\psi$. Herewith
$$\int K \circ d s = 2k \int \operatorname{rot} w \circ d s = 2k f'(\psi) \int \operatorname{rot} \psi \circ ds =$$
$$= 2k f'(\psi) \int v \circ d s.$$

the second order, then the thickness of the transition layer will be small in the first order, like the normal components of the velocity. The lateral pressure differences can be disregarded, as likewise any curvature of the streamlines. The pressure distribution will be impressed on the transition layer by the free fluid.

For the problem which has thus far been discussed, we obtain in the steady condition (X-direction tangential, Y-direction normal, u and v the corresponding velocity components) the differential equation

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{dp}{dx} = k \frac{\partial^2 u}{\partial y^2}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

If, as usual, dp/dx is given throughout, as also the course of u for the initial cross section, then every numerical problem of this kind can be numerically solved, by obtaining the corresponding $\partial u/\partial x$ by squaring every u. Thus we can always make progress in the X-direction with the aid of one of the well-known approximation methods (Cf. Kutta, "Zeitschrift für Math. und Physik," Vol. 46, p.435). One difficulty, however, consists in the various singularities developed on the solid surface. The simplest case of the conditions here considered is when the water flows along a flat thin plate. Here a reduction of the variables is possible and we can write $u = f\left(\frac{y}{\sqrt{x}} \right)$. By the numerical solution of the resulting differential equation, we obtain for the

drag the formula

$$R = 1.1 \ldots b\sqrt{k \rho l u_0^3}$$

(b width, l length of plate, u_0 velocity of undisturbed water opposite plate). Figure 1 shows the course of u.

The most important practical result of these investigations is that, in certain cases, the flow separates from the surface at a point entirely determined by external conditions (Fig. 2). A fluid layer, which is set in rotation by the friction on the wall, is thus forced into the free fluid and, in accomplishing a complete transformation of the flow, plays the same role as the Helmholz separation layers. A change in the viscosity constants k simply changes the thickness of the turbulent layer (proportional to the quantity $\sqrt{\dfrac{k l}{\rho u}}$, everything else remaining unchanged. It is therefore possible to pass to the limit k = 0 and still retain the same flow figure.

As shown by closer consideration, the necessary condition for the separation of the flow is that there should be a pressure increase along the surface in the direction of the flow. The necessary magnitude of this pressure increase in definite cases can be determined only by the numerical evaluation of the problem which is yet to be undertaken. As a plausible reason for the separation of the flow, it may be stated that, with a pressure increase, the free fluid, its kinetic energy is partially converted into potential energy. The transition layers, however, have lost a large part of their kinetic energy and no longer possess

enough energy to penetrate the region of higher pressure. They are
therefore deflected laterally.

According to the preceding, the treatment of a given flow
process is resolved into two components mutually related to one
another. On the one hand, we have the free fluid, which can be
treated as nonviscous according to the Helmholtz vortex laws,
while, on the other hand, we have the transition layers on the
solid boundaries, whose motion is determined by the free fluid,
but which, in their turn, impart their characteristic impress to
the free flow by the emission of turbulent layers.

I have attempted, in a few cases, to illustrate the process
more clearly by diagrams of the streamlines, though no claim is
made to quantitative accuracy. In so far as the flow is vortex-
free, one can, in drawing, take advantage of the circumstance,
that the streamlines form a quadratic system of curves with the
lines of constant potential.

Figures 3-4 show, in two stages, the beginning of the flow
around a wall projecting into the current. The vortex-free ini-
tial flow is rapidly transformed by a spiral separating layer.
The vortex continually advances, leaving still water behind the
finally stationary separating layer.

Figures 5-6 illustrate the analogous process with a cylinder.
The fluid layers set in rotation by the friction are plainly in-
dicated. Here also the separating layers extend into infinity.
All these separating layers are labile. If a slight sinoidal

disturbance is present, motions develop as shown in Figures 7-8. It is clearly seen how separate vortices are developed by the mutual interference of the flows. The vortex layer is rolled up inside these vortices, as shown in Figure 9. The lines of this figure are not streamlines, but such as were obtained by using a colored liquid.

I will now briefly describe experiments which I undertook for comparison with the theory. The experimental apparatus (Fig. 10) consists of a tank 1.5 m (nearly 5 feet) long with an intermediate bottom. The water is set in motion by a paddle wheel and, after passing through the deflecting apparatus a and four sieves b, enters the upper channel comparatively free from vortices, the object to be tested being introduced at c. Fine scales of micaceous iron ore are suspended in the water. These scales indicate the nature of the flow, especially as regards the vortices, by the peculiarities of their reflection due to their orientation.

The accompanying photographs were obtained in this manner, the flow being from left to right. Nos. 1-4 show the flow past a wall projecting into the current. The separating or boundary layer, which passes off from the edge, is apparent. In No. 1 it is very small; in No. 2, concealed by strong disturbances; in No. 3, the vortex spreads over the whole picture; in No. 4, the permanent condition is shown. A disturbance is also evident above the wall. Since a higher pressure prevails in the corner, due

to the obstruction of the water flow, even here the flow sepa-
rates from the wall after awhile (Cf. Figs. 1-4). The various
striae visible in the vortex-free portion of the flow (especially
in Nos. 1-2) are due to the fact that, at the inception of the
flow, the liquid was not entirely quiet. Nos. 5-6 show the flow
around a curved obstacle or, from another viewpoint, through a
continuously narrowing and then widening channel. No. 5 was taken
shortly after the inception of the flow. One boundary layer has
developed into a spiral, while the other has elongated and broken
up into very regular vortices. On the convex side, near the
right end, the beginning of the separation can be seen. No. 6
shows the permanent condition in which the flow begins to sepa-
rate about at the narrowest cross section.

Nos. 7-10 show the flow around a cylindrical obstacle. No.
7 shows the beginning of the separation; Nos. 8-9, subsequent
stages. Between the two vortices there is a line of water which
belonged to the transition layer before the beginning of the sep-
aration. No. 10 shows the permanent condition. The wake of tur-
bulent water behind the cylinder swings back and forth, whence
the momentary unsymmetrical appearance. The cylinder has a slot
along one of its generatrices. If this is placed as shown in
Nos. 11-12 and water is drawn out through a tube, the transition
layer on one side can be intercepted. When this is missing, its
effect, the separation, is eliminated. In No. 11, which corresponds,
in point of time, to No. 9, there is seen only one vortex and the
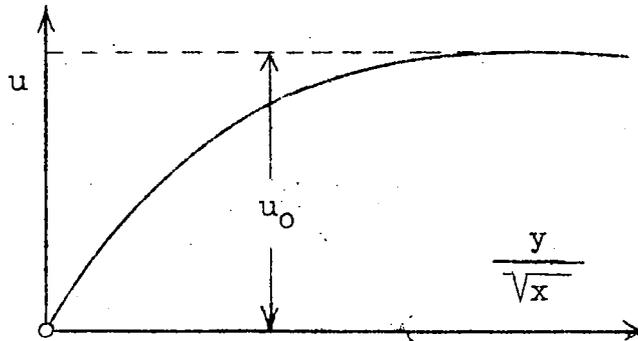
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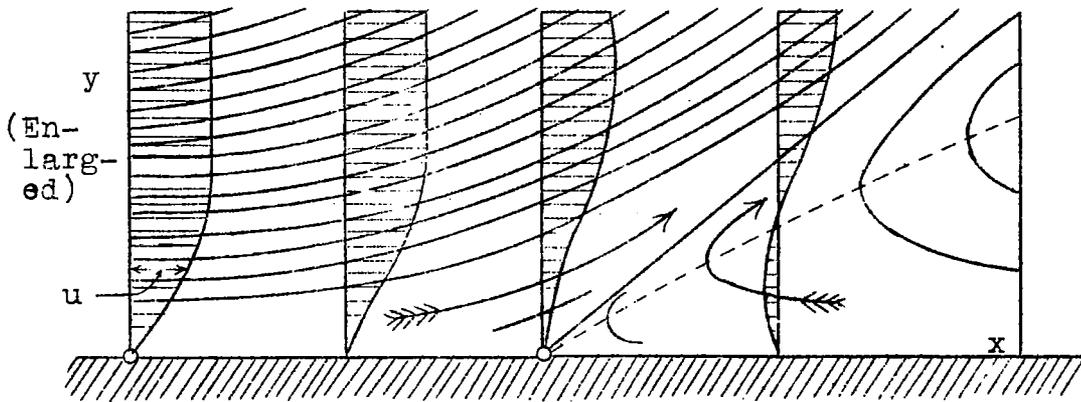
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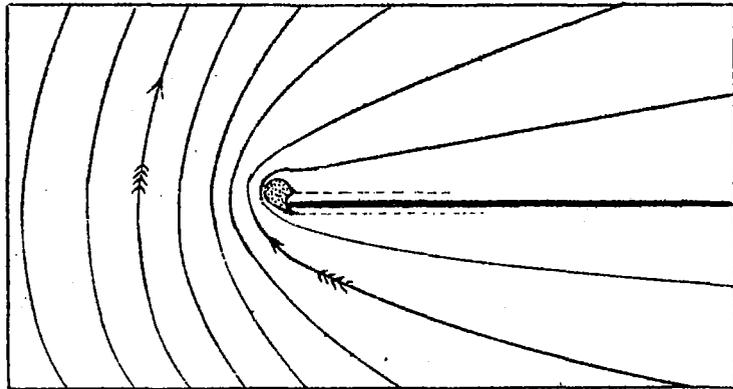
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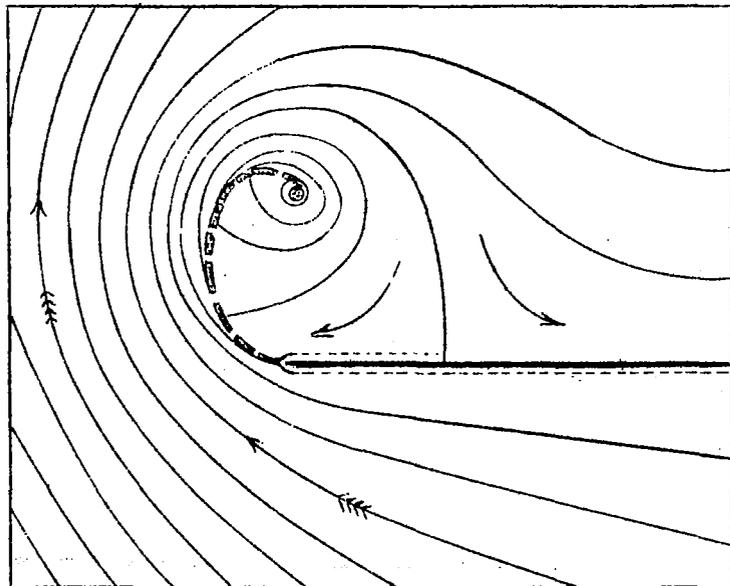
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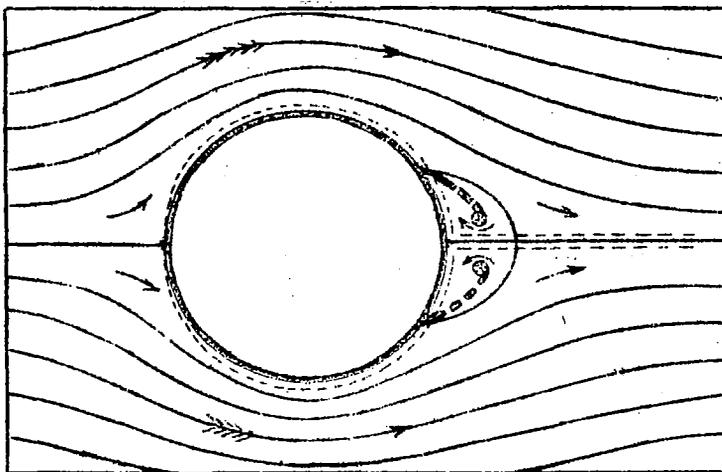
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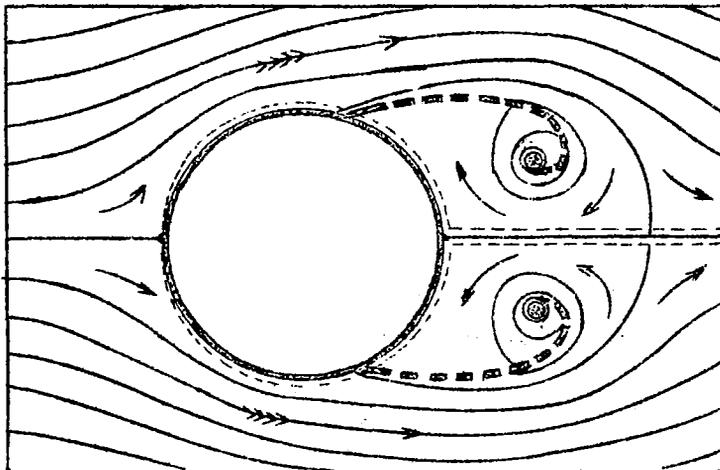
Fig.6

Fig.7

Fig.8

Fig.9

Fig.10